Lecture 29, Nov 19, 2025

Lyapunov Stability

- Consider a general nonlinear system $\dot{x} = f(x)$ where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is continuously differentiable
- An *equilibrium* state $\bar{x} \in \mathbb{R}^n$ is a state where $f(\bar{x}) = 0$, i.e. $x(t) = \bar{x}, \forall t \geq 0$

Definition

An equilibrium is *stable* if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\|x(0)\| < \delta \implies \|x(t)\| < \varepsilon, \forall t \geq 0$.

- In other words, given any positive radius ε around the equilibrium, we can find another radius δ, where if the initial state starts within δ of the equilibrium, then it will remain within a distance of ε of the equilibrium forever
  - This is the formal definition, which is very hard to work with, so in practice we use equivalent statements of the definition

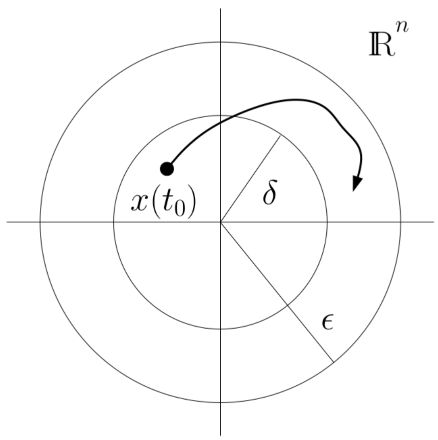

Figure 1: Illustration of the definition of Lyapunov stability.

Definition

An equilibrium is *asymptotically stable* if it is stable, and

$$\exists \delta_0 > 0 \text{ s.t. } \|x(0)\| < \delta_0 \implies \lim_{t \to \infty} x(t) = 0$$

- In other words, there exists a positive radius δ_0 such that if we start within this distance of the equilibrium, we always converge to the equilibrium
- A continuously differentiable function $U : \mathbb{R}^n \mapsto \mathbb{R}$ is *positive definite* at $x = 0$ if $U(0) = 0$ and $U(x) > 0$ for all $x \neq 0$; *negative definite* if $-U(x)$ is positive definite at 0 (i.e. $\forall x \neq 0, U(x) < 0$ and $U(0) = 0$)

Theorem

Lyapunov's Theorem: For a system where $f(0) = 0$, if we can find a continuously differentiable *Lyapunov function* $V : \mathbb{R}^n \mapsto \mathbb{R}$ which is positive definite at $x = 0$, then:

1. If $\dot{V}(x) = \dfrac{\partial V}{\partial x} f(x) \leq 0, \forall x \in \mathbb{R}$, then $x = 0$ is stable.
2. If $\dot{V}(x) = \dfrac{\partial V}{\partial x} f(x)$ is negative definite, then $x = 0$ is asymptotically stable.

- The Lyapunov function can be thought of as a measure of energy

- Consider a solution $x(t)$, then if $\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x}f(x) \leq 0$, then this "energy" either stays the same or decreases to zero
- If $V(x)$ is positive definite, then $V(x) = 0 \implies x = 0$, so as $V(x)$ converges to 0, x must also converge to its equilibrium
- We essentially converted all the high-dimensional dynamics of the system in x to the dynamics of a single scalar measure $V(x)$