Lecture 26, Nov 12, 2025

Modelling Example

- Example: Pendulum on a cart: a cart with mass m_1 is accelerated by a horizontal force u; there is a pendulum of mass m_2 attached to the cart, where the centre of mass is l units away from the pivot; the inertia of the pendulum about its centre of mass is I_{zz} (along the axis the pendulum will swing)
  - We can model this as a manipulator with a prismatic joint (for the cart) and revolute joint (for the pendulum)
    - According to DH frame rules, z_0 points in the direction of the cart track and z_1 points along the axis of rotation of the pendulum
  - Let x be the position of the cart and ϕ be the angle of the pendulum from vertical
    - In the DH table, $x = d_1$ and $\phi = \theta_2$
  - DH table:

Link	a	α	d	θ
1	0	$-\pi/2$	x	0
1	a_2	0	0	ϕ

  - Assume $r_1^0 = O_1^0$, so now we can write r_1^0, r_2^0 in terms of the generalized coordinates x and ϕ
    - $r_1^0 = \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix} \implies \dot{r}_1^0 = \begin{bmatrix} 0 \\ 0 \\ \dot{x} \end{bmatrix}$
    - $r_2^0 = \begin{bmatrix} l\cos\phi \\ 0 \\ x - l\sin\phi \end{bmatrix} \implies \dot{r}_2^0 = \begin{bmatrix} -l\sin(\phi)\dot{\phi} \\ 0 \\ \dot{x} - l\cos(\phi)\dot{\phi} \end{bmatrix}$
  - First find the kinetic energy
    - $T_1 = \frac{1}{2}m_1\|\dot{r}_1^0\|^2 = \frac{1}{2}m_1\dot{x}^2$ for link 1 since it is not rotating
    - $T_2 = \frac{1}{2}m_2\|\dot{r}_2^0\|^2 + \frac{1}{2}(w_2^0)^T I_2 w_2^0$
      - $\|\dot{r}_2^0\|^2 = (l\sin(\phi)\dot{\phi})^2 + (\dot{x} - l\cos(\phi)\dot{\phi})^2$
      - $w_2^0 = w_1^0 + R_1^0 w_2^1 = R_1^0 w_2^1 = R_1^0\dot{\phi}z_1^1 = \dot{\phi}z_1^0 = \begin{bmatrix} 0 \\ \dot{\phi} \\ 0 \end{bmatrix}$
      - $I_2 = R_2^0 \bar{I}_2 (R_2^0)^T$
      - $(R_2^0)^T w_2^0 = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ 0 & 0 & 1 \\ \sin\phi & -\cos\phi & 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ \dot{\phi} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}$
      - $(w_2^0)^T I_2 w_2^0 = \begin{bmatrix} 0 & 0 & \dot{\phi} \end{bmatrix} \bar{I}_2 \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} = I_{zz}\dot{\phi}^2$
    - $T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left((l\sin(\phi)\dot{\phi})^2 + (\dot{x} - l\cos(\phi)\dot{\phi})^2\right) + \frac{1}{2}I_{zz}\dot{\phi}^2$
      $= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}(I_{zz} + m_2 l^2)\dot{\phi}^2 - m_2 l\cos(\phi)\dot{x}\dot{\phi}$
      - $m = m_1 + m_2$ is the total mass
  - Now for the potential energy
    - $\bar{g} = \begin{bmatrix} -g \\ 0 \\ 0 \end{bmatrix}$
    - $\mathcal{U}(q) = -(m_1\bar{g}^T r_1^0 + m_2\bar{g}^T r_2^0) = m_2 g l\cos\phi$
  - Lagrangian: $\mathcal{L} = T - \mathcal{U} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}(I_{zz} + m_2 l^2)\dot{\phi}^2 - m_2 l\cos(\phi)\dot{x}\dot{\phi} - m^2 g l\cos\phi$

- Find derivatives... (exercise to the reader)
- Substitute into Euler-Lagrange to get the equations of motion:
  * $m\ddot{x} - m_2 l \cos(\phi)\ddot{\phi} + m_2 l \sin(\phi)\dot{\phi}^2 = u$
  * $(I_{zz} + m_2 l^2)\ddot{\phi} - m_2 l \cos(\phi)\ddot{x} - m_2 l g \cos(\phi) = 0$
- Put this in canonical form $D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau$
  * $q = \begin{bmatrix} x \\ \phi \end{bmatrix}$
  * $D(q) = \begin{bmatrix} m & -m_2 l \cos\phi \\ -m_2 l \cos\phi & I_{zz} + m_2 l^2 \end{bmatrix}$
    - Notice that, as expected, $D(q)$ is symmetric (if not, we made a calculation mistake!)
  * $G(q) = \begin{bmatrix} 0 \\ -m_2 l g \cos\phi \end{bmatrix}$
  * $C(q, \dot{q}) = \begin{bmatrix} 0 & m_2 l \sin(\phi)\dot{\phi} \\ 0 & 0 \end{bmatrix}$
    - This is the only term that's left
    - Notice the nonzero entry only has $\dot{\phi}$ instead of $\dot{\phi}^2$ since in the canonical form we multiply it again by $\dot{q}$ in the equation