Lecture 25, Nov 10, 2025

Canonical Robot Dynamics Model

- In our previous formula for kinetic energy, our $I_i(q)$ was expressed in the inertial frame, making it a function of q; we can instead write it in a body-fixed frame, which makes it constant, and relate it to the inertial frame inertia tensor via a similarity transform
  - $I_i = R_i^0 \bar{I}_i (R_i^0)^T$ where $\bar{I}_i$ is a constant inertia matrix expressed in a frame centred at the COM of link i
- Using this, $D(q) = \sum_{i=1}^{n} m_i J_{v_i}^T(q) J_{v_i}(q) + J_{w_i}^T(q) R_i^0 \bar{I}_i (R_0^i)^T J_{w_i}^T(q)$
  - $T = \frac{1}{2}\dot{q}^T D(q)\dot{q} = \frac{1}{2}\sum_{i,j}^{n} d_{ij}(q)\dot{q}_i\dot{q}_j$
  - Note $d_{ij}(q) = d_{ji}(q)$ due to symmetry
- $\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_k} = \sum_{j=1}^{n} d_{kj}(q)\dot{q}_j$
  - Note the $\frac{1}{2}$ disappears due to the symmetry of $D(q)$, so each term gets summed twice
- $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) &= \sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{j=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t}\left(d_{kj}(q)\right)\dot{q}_j \\ &= \sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{i,j} \frac{\partial d_{kj}(q)}{\partial q_i}\dot{q}_i\dot{q}_j\end{aligned}$$
- $\frac{\partial \mathcal{L}}{\partial q_k} = \frac{1}{2}\sum_{j=1}^{n} \frac{\partial d_{ij}(q)}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial \mathcal{U}}{\partial q_k}$
- Substitute into Euler-Lagrange: $\sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{i,j}\left(\frac{\partial d_{kj}(q)}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}(q)}{\partial q_k}\right)\dot{q}_i\dot{q}_j + \frac{\partial \mathcal{U}}{\partial q_k} = \tau_k$
  - It can be shown that $\sum_{i,j} \frac{\partial d_{kj}(q)}{\partial q_i}\dot{q}_i\dot{q}_j = \frac{1}{2}\left(\frac{\partial d_{kj}(q)}{\partial q_i} + \frac{\partial d_{ki}(q)}{\partial q_j}\right)\dot{q}_i\dot{q}_j$
- Let $c_{ijk} = \frac{1}{2}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right)$, then $\sum_j d_{kj}(q)\ddot{q}_j + \sum_{i,j} c_{ijk}\dot{q}_i\dot{q}_j + \frac{\partial \mathcal{U}}{\partial q_k} = \tau_k$
  - These are known as the *Christoffel symbols* (of the first kind)
- Organized in matrix form, $D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q)$
  - $D(q) \in \mathbb{R}^{n\times n}$ is the "mass matrix" for kinetic energy; this is the inertial term
  - $C(q,\dot{q}) \in \mathbb{R}^{n\times n}$ where $[C(q,\dot{q})]_{kj} = \sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i$ contains the Coriolis and centrifugal forces
  - $G(q) = \nabla_q \mathcal{U}(q) \in \mathbb{R}^n$ contains the forces due to gravity (or more generally, a potential)
- In general, for a robot modelling problem, we have 2 approaches:
  1. Using Euler-Lagrange: Writing out the kinetic and potential energies, computing the derivatives and substituting into the Euler-Lagrange equation
     - $T = \sum_{i=1}^{n} \frac{1}{2} m_i \|\dot{r}_i^0\|^2 + \frac{1}{2}(w_i^0)^T R_i^0 \bar{I}_i (R_0^i)^T (w_i^0)$
     - $\mathcal{U} = -\sum_{i=1}^{n} m_i \bar{g}_i^T r_i^0$
     - This is suitable for simple problems, where we can often obtain the r_i^0 by inspection and differentiate them
  2. Find $T = \frac{1}{2}\dot{q}^T D(q)\dot{q}$, then $C(q,\dot{q})$ using the formulas (which require $D(q)$ to be known), then

$$G(q) = \left(\frac{\partial \mathcal{U}}{\partial q}\right)^T$$

- This will work for any system

Summary

The canonical robot dynamics model is given by

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau$$

where the terms contain the inertial, Coriolis/centrifugal, and gravitational terms respectively:

- $D(q) = \sum_{i=1}^{n} \left(m_i J_{v_i}^T(q) J_{v_i}(q) + J_{w_i}^T(q) R_i^0 \bar{I}_i (R_0^i)^T J_{w_i}^T(q) \right)$
- $[C(q, \dot{q})]_{kj} = \sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i$ where $c_{ijk} = \frac{1}{2}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right)$
- $G(q) = \nabla_q \mathcal{U}(q)$ where $\mathcal{U}(q) = -\sum_{i=1}^{n} m_i \bar{g}^T r_i^0(q)$

where $\bar{I}_i$ are the inertias of each link measured about its centre of mass in a body-fixed frame, r_i^0 are the centres of mass, $\bar{g} = \begin{bmatrix} 0 & 0 & -g \end{bmatrix}^T$ points in the direction of gravitational acceleration, and the Jacobians are given by

$$J_{v_i} = \begin{cases} \begin{bmatrix} z_0^0 \times (r_i^0) & z_1^0 \times (r_i^0 - O_1^0) & \cdots & z_{i-1}^0 \times (r_i^0 - O_{i-1}^0) & 0_{3\times 3(n-i)} \end{bmatrix} & \text{joint } i \text{ is revolute} \\ \begin{bmatrix} z_0^0 & z_1^0 & \cdots & z_{i-1}^0 & 0_{3\times 3(n-i)} \end{bmatrix} & \text{joint } i \text{ is prismatic} \end{cases}$$

$$J_{w_i} = \begin{bmatrix} \rho_1 z_0^0 & \rho_2 z_1^0 & \cdots & \rho_i z_{i-1}^0 & 0_{3\times 3(n-i)} \end{bmatrix}$$