Lecture 23, Nov 5, 2025

Euler-Lagrange – Part 3

- *Kinetic energy* for a collection of point masses is $T = \sum_{i=1}^{N} \frac{1}{2} m_i \|\dot{r}_i\|^2 = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{r}_i^T \dot{r}_i$
- Notice
$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) &= \sum_{i=1}^{N} \frac{\mathrm{d}}{\mathrm{d}t}\left(m_i \dot{r}_i^T \frac{\partial \dot{r}_i}{\partial \dot{q}_j}\right) \\ &= \sum_{i=1}^{N} \frac{\mathrm{d}}{\mathrm{d}t}\left(m_i \dot{r}_i^T \frac{\partial r_i}{\partial q_j}\right) \end{aligned}$$
  - Note $\dot{r}_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \dot{q}_j$, so $\frac{\partial \dot{r}_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}$
  - This is the first term in the summation for $\sum_{i=1}^{N} m_i \ddot{r}_i^T \delta r_i$ from last lecture
  - Also, $\frac{\partial T}{\partial q_j} = \sum_{i=1}^{N} m_i \dot{r}_i^T \frac{\partial \dot{r}_i}{\partial q_j}$ which is the second term
- Combining everything: $\sum_{j=1}^{n} \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right) \mathrm{d}q_j = \sum_{j=1}^{n} \varphi_j \, \mathrm{d}q_j$
  - Since the $\mathrm{d}q_j$ are arbitrary, this means $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = \varphi_j, j = 1, \dots, n$
- For each force, $f_i^l = f_i^g + f_i^a = -\nabla_{r_i} \mathcal{U}(r_1, \dots, r_N) + f_i^a = -\frac{\partial \mathcal{U}}{\partial r_i} + f_i^a$, where the first term is the force caused by gravity
  - $$\begin{aligned} \varphi_j &= \sum_{i=1}^{N} (f_i^l)^T \frac{\partial r_i}{\partial q_j} \\ &= \sum_{i=1}^{N} -\frac{\partial \mathcal{U}}{\partial r_i} \frac{\partial r_i}{\partial q_j} + (f_i^a)^T \frac{\partial r_i}{\partial q_j} \\ &= -\frac{\partial \mathcal{U}}{\partial q_j} + \tau_j \end{aligned}$$
  - The term τ_j captures all the other forces acting on the joint
- Finally, we get $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial \mathcal{U}}{\partial q_j} = \tau_j$
- Let the *Lagrangian* $\mathcal{L} = T - \mathcal{U}$
  - Note the potential energy is independent of $\dot{q}_j$, so $\frac{\partial T}{\partial \dot{q}_j} = \frac{\partial \mathcal{L}}{\partial \dot{q}_j}$
- We arrive at the *Euler-Lagrange equations*: $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = \tau_j$

Summary

The Euler-Lagrange equations are a set of n equations of motion for the system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = \tau_j$$

where the Lagrangian is $\mathcal{L} = T - \mathcal{U}$, n is the number of degrees of freedom ($n = 3N - l$ for l constraints and N particles in 3 dimensions), q are n generalized coordinates which parametrize the set of allowed states, τ_j are the generalized forces:

$$\tau_j = \sum_{i=1}^{N} (f_i^l)^T \frac{\partial r_i}{\partial q_j}$$

where f^l are the applied forces.
A set of l independent holonomic constraints are expressed as

$$g(r_1, \ldots, r_N) = 0 \in \mathbb{R}^l \qquad \text{rank}\left(\frac{\partial g}{\partial r}\right) = l$$

For constraints to be satisified, the virtual displacements $\delta r = \begin{bmatrix} \delta r_1^T & \cdots & \delta r_N^T \end{bmatrix}^T \in \mathbb{R}^{3N}$ must satisfy $\frac{\partial g}{\partial r} \delta r = 0$.