Lecture 22, Nov 3, 2025

Euler-Lagrange – Part 2

- Last lecture we derived the Lagrange-d'Alembert principle: $\sum_{i=1}^{N}(m_i\ddot{r}_i - f_i^l)^T\delta r_i = 0$
  - To get to the form we want, we'll get rid of all the r terms and express everything in terms of q
- From the chain rule:
  - $\dot{r}_i = \sum_{j=1}^{n}\frac{\partial r_i}{\partial q_j}\dot{q}_j$
  - $\delta r_i = \sum_{j=1}^{n}\frac{\partial r}{\partial q_j}\mathrm{d}q_j$
    - Note here $\mathrm{d}q_j$ is not a virtual displacement, since unlike δr_i it is not constrained
- The two parts of the equation are then:
  - $$\begin{aligned}\sum_{i=1}^{N}(f_i^l)^T\delta r_i &= \sum_{i=1}^{N}\sum_{j=1}^{n}(f_i^l)^T\frac{\partial r_i}{\partial q_j}\mathrm{d}q_j\\ &= \sum_{j=1}^{n}\varphi_j\mathrm{d}q_j\end{aligned}$$
    - $\varphi_j = \sum_{i=1}^{N}(f_i^l)^T\frac{\partial r_i}{\partial q_j}$ is the jth *generalized force*
  - $\sum_{i=1}^{N}m_i\ddot{r}_i^T\delta r_i = \sum_{i=1}^{N}\sum_{j=1}^{n}m_i\ddot{r}_i^T\frac{\partial r_i}{\partial q_j}\mathrm{d}q_j$
- Observe $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(m_i\dot{r}_i^T\frac{\partial r_i}{\partial q_j}\right) &= m_i\ddot{r}_i^T\frac{\partial r_i}{\partial q_j} + m_i\dot{r}_i^T\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial r_i}{\partial q_j}\right)\\ &= m_i\ddot{r}_i^T\frac{\partial r_i}{\partial q_j} + m_i\dot{r}_i^T\frac{\partial \dot{r}_i}{\partial q_j}\end{aligned}$$
  - $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial r_i}{\partial q_j}\right) = \sum_{k=1}^{n}\frac{\partial}{\partial q_k}\left(\frac{\partial r_i}{\partial q_j}\right)\dot{q}_k = \frac{\partial}{\partial q_j}\sum_{k=1}^{n}\frac{\partial r_i}{\partial q_k}\dot{q}_k = \frac{\partial \dot{r}_i}{\partial q_j}$
- Therefore $\sum_{i=1}^{N}m_i\ddot{r}_i^T\delta r_i = \sum_{i=1}^{N}\sum_{j=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(m_i\dot{r}_i^T\frac{\partial r_i}{\partial q_j}\right) - m_i\dot{r}_i^T\frac{\partial \dot{r}_i}{\partial q_j}\right)\mathrm{d}q_j$