Lecture 21, Oct 24, 2025

Euler-Lagrange – Part 1

- We will develop a model for the dynamics of our manipulator, so we can do higher fidelity control
  - Our goal is to derive the model $M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau$ presented in the last lecture, which is the standard manipulator model in robotics
- Consider N point masses in $\mathbb{R}^3$, and let $r_i \in \mathbb{R}^3$ denote the position of mass i
  - Each mass satisfies its equations of motion, $m_i\ddot{r}_i - f_i^l - f_i^c = 0$, where f_i^l is the sum of load forces for link i and f_i^c is the sum of constraint forces for link i (i.e. forces that hold the links together)
  - The masses are subject to a set of l holonomic constraints, i.e. their positions are constrained with respect to each other
    * The constraints are expressed as $g(r_1, \dots, r_N) = 0$ where $g : \mathbb{R}^3 \times \cdots \times \mathbb{R}^3 \mapsto \mathbb{R}^l$ (this is a vector valued function, since we stack all l constraints)
    * To enforce independence, assume that $\dfrac{\partial g}{\partial r} \in \mathbb{R}^{l\times 3N}$ is such that $\operatorname{rank}\left(\dfrac{\partial g}{\partial r}\right) = l$, i.e. the Jacobian is full-rank
  - Let $n = 3N - l$ be the degrees of freedom of the system after the constraints are accounted for
    * Assume we have identified n *generalized coordinates*, $(q_1, \dots, q_n)$, which parametrize the degrees of freedom of the system
    * These turn out to be the exact same as the joint variables
  - The set of allowed states is $\Gamma = \{\, (r_1, \dots, r_N) \mid g(r_1, \dots, r_N) = 0 \,\}$, which is parametrized by the n generalized coordinates, i.e. there is a one-to-one mapping between $(q_1, \dots, q_n)$ and elements of Γ
    * Written explicitly, $r_i = r_i(q_1, \dots, q_n)$
  - Practically, for a robot, each r_i is taken at the centre of mass of the link (instead of at O_i), so the dynamics work out
- Consider an N-link planar manipulator with all revolute joints
  - For our first link r_1, we have 2 constraints – it stays in the xy plane, and its position on the link stays the same, i.e. its distance from O_0 stays the same
    * The planar constraint is linear, but the distance constraint would be nonlinear since we need to square $r_{i,x}$ and $r_{i,y}$
  - So in total, the number of constraints here is $l = 2N$
  - So our degrees of freedom is $n = 3N - l = 3N - 2N = N$, which is the same as the number of links as we expected
- Let $\delta r = \begin{bmatrix}\delta r_1^T & \cdots & \delta r_N^T\end{bmatrix}^T \in \mathbb{R}^{3N}$ be a *virtual displacement*, i.e. a "virtual" small perturbation
  - We require $\displaystyle\sum_{j=1}^{n} \frac{\partial g}{\partial r_j}\delta r_j = \frac{\partial g}{\partial r}\delta r = 0$ – notice the similarity to the chain rule
    * This can be derived by differentiating $g(r(t)) = 0$ and noting that the result has to be zero for all t
  - Intuitively, this means that if we have infinitesimal perturbations to the point masses, we need the perturbations to satisfy all the constraints; i.e. any movement should stay in Γ by staying on the level set $g(r) = 0$
- Starting from the equations of motion, we have $(m_i\ddot{r}_i - f_i^l - f_i^c)^T\delta r_i = 0$
  - The constraint forces only act in directions orthogonal to the allowable directions (they have no impact in the allowable directions), i.e. $(f_i^c)^T\delta r_i = 0$, so we can ignore the f_i^c term above
  - Over all i, $\displaystyle\sum_{i=1}^{N}(m_i\ddot{r}_i - f_i^l)^T\delta r_i = 0$
  - This is known as the *Lagrange-d'Alembert principle*