Lecture 18, Oct 15, 2025

Obstacle Avoidance via Potential Field

- Given a start joint position q^s and end position q^f, the motion planning problem involves determining the path of the manipulator in q-space, $q(t)$, such that the robot doesn't hit any obstacles
  - Denote by $\bar{O}_i^0$ the final position of the base point O_i^0 at q_f
- We will use a potential field approach, with $\mathcal{U}(q) = \mathcal{U}_{att}(q) + \mathcal{U}_{rep}(q)$
  - $\mathcal{U}_{att}(q)$: the attractive potential function, which pulls each O_i^0 to $\bar{O}_i^0$
    - This potential alone would simply pull each base point in a straight line to their final positions
  - $\mathcal{U}_{rep}(q)$: the repulsive potential function, which pushes each O_i^0 away from obstacles
    - The strength of this potential is based on the distance between the base points and the nearest point on an obstacle, i.e. the orthogonal projection
- Note this assumes the obstacles are convex, otherwise looking at only the base points is not enough to avoid the obstacle (with some other assumptions, e.g. obstacles are not too small, etc)
  - More complex versions of this algorithm can use various heuristics or intermediate waypoints, etc
  - In general this problem is computationally intractable (NP-hard)
    - Potential fields are not guaranteed to always solve the problem but they are easy to compute
- For now we work in Euclidean space instead of q-space, but in the end we need to define potentials in q
- Define the attractive potential $\mathcal{U}_{i,att}(O_i^0) = \dfrac{1}{2}c_i\|O_i^0 - \bar{O}_i^0\|^2$ where $c_i > 0$ is a constant weight
  - The weights allow us to place different importances for the different base points, effectively having some of them converge faster than the others
- Attractive gradient:

$$\begin{aligned}F_{i,att}(O_i^0) &= -\nabla\mathcal{U}_{i,att}(O_i^0)\\ &= -\left(\frac{\partial\mathcal{U}_{i,att}(O_i^0)}{\partial O_i^0}\right)^T\\ &= -\left(c_i\|O_i^0 - \bar{O}_i^0\| \cdot \frac{(O_i^0 - \bar{O}_i^0)^T}{\|O_i^0 - \bar{O}_i^0\|}\right)^T\\ &= -c_i(O_i^0 - \bar{O}_i^0)\end{aligned}$$

  - Notice how this directly pulls towards $\bar{O}_i^0$ in a straight line, with a strength proportional to the distance times a weight
  - Practically, we want to avoid very large gradients, so we often cap the magnitude of the gradient at some value, or normalize the gradient past a certain point
  - $F_{i,att}(O_i^0) = \begin{cases} -c_i(O_i^0 - \bar{O}_i^0) & \|O_i^0 - \bar{O}_i^0\| \le d \\ -c_i\dfrac{(O_i^0 - \bar{O}_i^0)}{\|O_i^0 - \bar{O}_i^0\|}d & \|O_i^0 - \bar{O}_i^0\| > d \end{cases}$
- Define the repulsive potential $\mathcal{U}_{i,rep} = \begin{cases} \dfrac{\eta_i}{2}\left(\dfrac{1}{\|O_i^0 - \pi(O_i^0)\|} - \dfrac{1}{\rho_0}\right)^2 & \|O_i^0 - \pi(O_i^0)\| \le \rho_0 \\ 0 & \|O_i^0 - \pi(O_i^0)\| > \rho_0 \end{cases}$
  - $\mathcal{O}$ is a convex set representing an obstacle ($\forall p, q \in \mathcal{O}, \overline{pq} \in \mathcal{O}$) in frame 0, and $\pi(p)$ is the orthogonal projection of p onto $\mathcal{O}$
    - Note for all convex $\mathcal{O} \in \mathbb{R}^3$, for each $p \in \mathbb{R}^3$ where $p \notin \mathcal{O}$, there exists a unique $\pi(p) \in \mathcal{O}$ such that $\|p - \pi(p)\|$ is minimum, i.e. there is a unique closest point in $\mathcal{O}$ to p
    - The line joining p and $\pi(p)$ is orthogonal to the boundary of $\mathcal{O}$
  - η_i is another weight parameter similar to in the attractive potential
  - ρ_0 is some distance threshold; if the base point is outside this threshold, the obstacle has no effect
    - $\{\, O_i^0 \mid \|O_i^0 - \pi(O_i^0)\| \le \rho_0 \,\}$ is the *region of influence* of $\mathcal{O}$ for collision avoidance
  - We've construed the repulsive potential so that it is continuous, and $\mathcal{U}_{i,rep} \to \infty$ as $O_i^0 \to \pi(O_i^0)$, i.e. the repulsion is stronger the closer we are to the obstacle
- Note the columns of $\dfrac{\partial\pi}{\partial O_i^0}$ are orthogonal to $O_i^0 - \pi(O_i^0)$, i.e. $(O_i^0 - \pi(O_i^0))^T\dfrac{\partial\pi}{\partial O_i^0} = 0$, due to the geometry of the orthogonal projection

- Intuitively $\frac{\partial \pi}{\partial O_i^0}$ is a line tangent to the boundary of $\mathcal{O}$, so it should be orthogonal to the line $O_i^0 - \pi(O_i^0)$

- $$\begin{aligned}\frac{\partial \mathcal{U}_{i,rep}(O_i^0)}{\partial O_i^0} &= -\eta_i \left(\frac{1}{\|O_i^0 - \pi(O_i^0)\|} - \frac{1}{\rho_0} \right) \frac{1}{\|O_i^0 - \pi(O_i^0)\|^2} \frac{(O_i^0 - \pi(O_i^0))^T}{\|O_i^0 - \pi(O_i^0)\|} \frac{\partial (O_i^0 - \pi(O_i^0))}{\partial O_i^0} \\ &= -\eta_i \left(\frac{1}{\|O_i^0 - \pi(O_i^0)\|} - \frac{1}{\rho_0} \right) \frac{(O_i^0 - \pi(O_i^0))^T}{\|O_i^0 - \pi(O_i^0)\|^3} \left(I - \frac{\partial \pi(O_i^0)}{\partial O_i^0} \right) \\ &= -\eta_i \left(\frac{1}{\|O_i^0 - \pi(O_i^0)\|} - \frac{1}{\rho_0} \right) \frac{(O_i^0 - \pi(O_i^0))^T}{\|O_i^0 - \pi(O_i^0)\|^3}\end{aligned}$$

  - $F_{i,rep}(O_i^0) = \begin{cases} \eta_i \left(\frac{1}{\|O_i^0 - \pi(O_i^0)\|} - \frac{1}{\rho_0} \right) \frac{(O_i^0 - \pi(O_i^0))}{\|O_i^0 - \pi(O_i^0)\|^3} & \|O_i^0 - \pi(O_i^0)\| \leq \rho_0 \\ 0 & \|O_i^0 - \pi(O_i^0)\| > \rho_0 \end{cases}$