Lecture 17, Oct 10, 2025

End-Effector Force and Torque Problem

- Given a desired force f^0 and torque n^0 at the end-effector, what are the corresponding forces and torques we need to apply at the joints to produce this force and torque?
  - In physics, $F^0 = \begin{bmatrix} f^0 \\ n^0 \end{bmatrix} \in \mathbb{R}^6$ is known as a *wrench*
- Consider a body with linear velocity $v^0(t)$ and angular velocity $w^0(t)$ subject to a wrench F^0, then from physics the work performed by the wrench over $[t_1, t_2]$ is $W = \displaystyle\int_{t_1}^{t_2} (v^0(t)^T f^0(t) + w^0(t)^t n^0(t))\,\mathrm{d}t$
  - More compactly, $W = \displaystyle\int_{t_1}^{t_2} \xi^0(t)^T F^0(t)\,\mathrm{d}t$ where $\xi = \begin{bmatrix} v^0(t) \\ w^0(t) \end{bmatrix}$
  - Substitute $\xi = \begin{bmatrix} \dot{O}_n^0 \\ w_n^0 \end{bmatrix} = J(q)\dot{q}$ we get $W = \displaystyle\int_{t_1}^{t_2} \dot{q}(t)^T J(q(t))^T F^0(t)\,\mathrm{d}t$
- Now consider the total work done at each joint (assuming all revolute joints for now for simplicity): $W_r = \displaystyle\int_{t_1}^{t_2} \sum_{i=1}^{n} \dot{q}_i(t)\tau_i(t)\,\mathrm{d}t = \int_{t_1}^{t_2} \dot{q}(t)^T \tau(t)\,\mathrm{d}t$ where $\tau(t) = \begin{bmatrix} \tau_1(t) & \cdots & \tau_n(t) \end{bmatrix}^T \in \mathbb{R}^n$
- We can equate them due to energy conservation: $\displaystyle\int_{t_1}^{t_2} \dot{q}(t)^T \tau(t)\,\mathrm{d}t = \int_{t_1}^{t_2} \dot{q}(t)^T J(q(t))^T F^0(t)\,\mathrm{d}t$
  - Using the argument that $\dot{q}(t)$ is entirely arbitrary, we conclude $\tau(t) = J(q(t))^T F^0(t)$
  - Now we can use this formula to get the joint torques we need
- A typical application: given $F^{\text{load}} = \begin{bmatrix} f^{\text{load}} \\ n^{\text{load}} \end{bmatrix}$, ignoring the robot's own weight, we apply $F^0 = -F^{\text{load}}$ to counteract the load, so the torque we need to apply at each joint is $\tau = J(q)^T F^0$

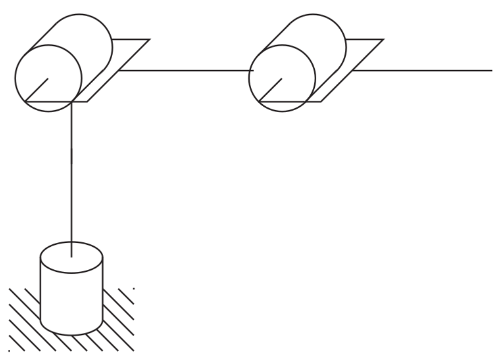

Figure 1: Robot for the torque example.

- Example: consider a 3-link RRR articulated robot; the end-effector must hold a mass M which applies a force $Mg = 10\,\text{N}$; ignoring the weight of the robot, what torques do we need to apply?
  - $F^{\text{load}} = \begin{bmatrix} 0 \\ 0 \\ -10 \\ 0 \\ 0 \\ 0 \end{bmatrix} \implies F^0 = \begin{bmatrix} 0 \\ 0 \\ 10 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

- $\tau = J(q)^T F^0$

  $$= \begin{bmatrix} z_0^0 \times O_3^0 & z_1^0 \times (O_3^0 - O_1^0) & z_2^0 \times (O_3^0 - O_2^0) \\ z_0^0 & z_1^0 & z_2^0 \end{bmatrix}^T F^0$$

  $$= \begin{bmatrix} (z_0^0 \times O_3^0)^T \\ (z_1^0 \times (O_3^0 - O_1^0))^T \\ (z_2^0 \times (O_3^0 - O_2^0))^T \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

  $$= \begin{bmatrix} 0 \\ 10a_3 \cos(\theta_2 + \theta_3) + 10a_2 \cos(\theta_2) \\ 10a_3 \cos(\theta_2 + \theta_3) \end{bmatrix}$$
- Notice how the first component is 0, which makes sense since the first joint does not contribute to holding up a load at all
- Note how the torque we need to apply depends on what the joint angles are
  * τ_2, τ_3 are maximum when $\cos(\theta_2 + \theta_3) = \cos(\theta_2) = 1$; intuitively this corresponds to the situation where the joints are fully extended and lined up
  * Conversely we get $\tau_2 = \tau_3 = 0$ if $\cos(\theta_2 + \theta_3) = \cos(\theta_2) = 0$, which intuitively corresponds to the situation where both joints are pointing straight up, so we don't need to apply any torque to hold up the load