Lecture 14, Oct 3, 2025

Forward Velocity Kinematics (Continued)

Linear Velocity Jacobian

- Now we want to find $J_v(q)$ such that $\dot{O}_n^0 = J_v(q)\dot{q}$, where $J_v(q) = \dfrac{\partial O_n(q)}{\partial q}$
  - Note for simple cases, if we have an explicit expression for the end-effector position as a function of q (from forward kinematics), we can simply differentiate it as a consequence of the chain rule
  - However we want a systematic approach from only the DH parameters so we can do this for any general manipulator
- Recall that $O_i^0 = O_{i-1}^0 + R_{i-1}^0 O_i^{i-1} \implies O_i^0 - O_{i-1}^0 = R_{i-1}^0 O_i^{i-1}$
  - Therefore $O_n^0 = \sum_{i=1}^{n} O_n^0 - O_{n-1}^0 = \sum_{i=1}^{n} R_{i-1}^0 O_i^{i-1} \implies \dot{O}_n^0 = \sum_{i-1}^{n}(\dot{O}_i^0 - \dot{O}_{i-1}^0)$
- Differentiate each term in the sum: $\begin{aligned}\dot{O}_i^0 - \dot{O}_{i-1}^0 &= \dot{R}_{i-1}^0 O_i^{i-1} + R_{i-1}^0 \dot{O}_i^{i-1} \\ &= S(w_{i-1}^0) R_{i-1}^0 O_i^{i-1} + R_{i-1}^0 \dot{O}_i^{i-1} \\ &= w_{i-1}^0 \times (R_{i-1}^0 O_i^{i-1}) + R_{i-1}^0 \dot{O}_i^{i-1}\end{aligned}$
  - For prismatic joints, $\dot{O}_i^{i-1} = \dot{q}_i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \dot{q}_i z_{i-1}^{i-1} \implies R_{i-1}^0 \dot{O}_i^{i-1} = \dot{q}_i R_{i-1}^0 z_{i-1}^{i-1} = \dot{q}_i z_{i-1}^0$
  - For revolute joints, $\dot{O}_i^{i-1} = \dot{q}_i z_{i-1}^{i-1} \times O_i^{i-1} \implies \begin{aligned}R_{i-1}^0 \dot{O}_i^{i-1} &= \dot{q}_i R_{i-1}^0 z_{i-1}^{i-1} \times R_{i-1}^0 O_i^{i-1} \\ &= (\dot{q}_i z_{i-1}^0) \times (R_{i-1}^0 O_i^{i-1}) \\ &= \dot{q}_i (z_{i-1}^0 \times (R_{i-1}^0 O_i^{i-1}))\end{aligned}$
  - Then $\dot{O}_i^0 - \dot{O}_{i-1}^0 = w_{i-1}^0 \times (R_{i-1}^0 O_i^{i-1}) + \dot{q}_i \begin{cases} z_{i-1}^0 & \text{joint } i \text{ is prismatic} \\ z_{i-1}^0 \times (R_{i-1}^0 O_i^{i-1}) & \text{joint } i \text{ is revolute} \end{cases}$
  - Using a similar derivation as the angular velocity Jacobian, $w_{i-1}^0 = \sum_{j=1}^{i-1} \dot{q}_j \rho_j z_{j-1}^0$
  - Finally, $\dot{O}_i^0 - \dot{O}_{i-1}^0 = \left(\sum_{j=1}^{i-1} \dot{q}_j \rho_j z_{j-1}^0 \right) \times (R_{i-1}^0 O_i^{i-1}) + \dot{q}_i \begin{cases} z_{i-1}^0 & \text{joint } i \text{ is prismatic} \\ z_{i-1}^0 \times (R_{i-1}^0 O_i^{i-1}) & \text{joint } i \text{ is revolute} \end{cases}$
- To collect everything, note the following:
  - If joint k is prismatic, then $\rho_k = 0$, and so $\dot{q}_k$ can only show up in the second term, and will have a coefficient of z_{i-1}^0
  - If joint k is revolute, then $\rho_k = 1$ and $\dot{q}_k$ shows up in both terms
    * $\dot{q}_k$ appears in each term $\dot{O}_i^0 - \dot{O}_{i-1}^0$ where $i \geq k$ due to the double sum
    * Intuitively, the revolute joint's angular velocity affects all links down the chain
    * Using this, we can show that the coefficient of $\dot{q}_k$ in $\dot{O}_n^0$ is $z_{k-1}^0 \times \left(\sum_{i=k}^{n} R_{i-1}^0 O_i^{i-1} \right)$
      - We also know $O_n^0 = O_{k-1}^0 + R_{k-1}^0 O_k^{k-1} + \cdots + R_{n-1}^0 O_n^{n-1}$ so $\sum_{i=k}^{n} R_{i-1}^0 O_i^{i-1} = O_n^0 - O_{k-1}^0$
- Finally, $J_v(q) = \begin{bmatrix} J_{v,1}(q) & \cdots & J_{v,n}(q) \end{bmatrix} \in \mathbb{R}^{3 \times n}$, where each column can be written as:
  - $J_{v,i}(q) = \begin{cases} z_{i-1}^0 & \text{joint } i \text{ is prismatic} \\ z_{i-1}^0 \times (O_n^0 - O_{i-1}^0) & \text{joint } i \text{ is revolute} \end{cases}$