Lecture 13, Oct 1, 2025

Forward Velocity Kinematics (Continued)

- Recall that we want to find a formula for the end-effector linear and angular velocity, so we want to find $\dot{O}_i^{i-1}, w_i^{i-1}$ in terms of the DH parameters
- Denote $w_{i,j}^k = R_i^k w_j^i$ as a shorthand (notation used by textbook)
  - This means "the angular velocity of frame j with respect to frame i, expressed in frame k"
  - Using this notation, $w_n^0 = w_{0,1}^0 + \cdots + w_{n-1,n}^0 = \sum_{i=1}^{n} w_{i-1,i}^0$
- Recall that rotating a body around a fixed axis z yields an angular velocity $w = \dot{\theta} z$ where $\dot{\theta}$ is the rate of rotation, and the linear velocity of any point on the object is given by $v = w \times p$, where p is a vector from the axis of rotation
- Therefore, we can derive $\dot{O}_i^{i-1}, w_i^{i-1}$ for each type of joint as follows:
  - Revolute joint: $w_i^{i-1} = \dot{q}_i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \dot{O}_i^{i-1} = w_i^{i-1} \times O_i^{i-1}$
    * $w_i = \dot{\theta}_i z_{i-1}$ since the joint rotates around axis z_{i-1} and the angle of rotation is θ
    * Then $w_i^{i-1} = \dot{\theta}_i z_{i-1}^{i-1} = \dot{\theta}_i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
    * From physics, $\dot{O}_i^{i-1} = w_i^{i-1} \times O_i^{i-1}$
      - In the common case where O_i and O_{i-1} are aligned on z_{i-1}, notice that this evaluates to zero (since z_{i-1} is parallel to w_i), which is consistent with our intuition that in this case, frame i is not translating with respect to frame $i-1$
  - Prismatic joint: $w_i^{i-1} = 0, \dot{O}_i^{i-1} = \dot{q}_i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
    * This can be easily seen by noting that there is no rotation happening
    * The only component of O_i^{i-1} that is changing is the z component, which has a value of d_i

Angular Velocity Jacobian

- We will show the following:
  - $\dot{O}_n^0 = J_v(q)\dot{q}$, where $J_v(q) = \dfrac{\partial O_n(q)}{\partial q}$ is the *linear velocity Jacobian*
  - $w_n^0 = J_w(q)\dot{q}$, where $J_w(q) = \dfrac{\partial w_n}{\partial q}$ is the *angular velocity Jacobian*
- For the angular velocity, recall $w_n^0 = w_{0,1}^0 + \cdots + w_{n-1,n}^0 = \sum_{i=1}^{n} w_{i-1,i}^0$
  - If joint i is revolute, then $w_{i-1,i}^0 = R_{i-1}^0 w_i^{i-1} = R_{i-1}^0 \dot{q}_i z_{i-1}^{i-1} = \dot{q}_i R_{i-1}^0 z_{i-1}^{i-1} = \dot{q}_i z_{i-1}^0$
  - If joint i is prismatic, then $w_{i-1,i}^0 = 0$
- As a shorthand, define the *indicator function* $\rho_i = \begin{cases} 0 & \text{joint } i \text{ is prismatic} \\ 1 & \text{joint } i \text{ is revolute} \end{cases}$
- Using the indicator function,
$$\begin{aligned} w_n^0 &= \sum_{i=1}^{n} w_{i-1,i}^0 \\ &= \rho_i \dot{q}_i z_{i-1}^0 \\ &= \begin{bmatrix} \rho_1 z_0^0 & \rho_1 z_1^0 & \cdots & \rho_n z_{n-1}^0 \end{bmatrix}_{3\times n} \dot{q} \\ &= J_w(q)\dot{q} \end{aligned}$$