Lecture 12, Sep 29, 2025

Forward Velocity Kinematics

- Given the joint variables $q_i(t)$ as functions of time, we want to find the relationship between the joint velocities $\dot{q}_i(t)$ and the linear and angular velocities of the end-effector, with applications in motion planning
  - To do this, we need to know how velocities are transformed between frames
- Define the operation $S(w) = S\left(\begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}\right) = \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix}$
  - This allows us to compute the cross product easily as $w \times v = S(w)v$
  - Note this produces a skew-symmetric matrix, i.e. $M = -M^T$; furthermore, every skew-symmetric matrix can be expressed as $S(w)$ for some unique $w \in \mathbb{R}^3$
- Note the following properties of $S(w)$:
  1. Linearity: $S(w_1 + \lambda w_2) = S(w_1) + \lambda S(w_2)$
  2. Similarity transform: $RS(w)R^T = S(Rw)$ for all $R \in SO(3)$
- Lemma: there exists a unique $w_1^0 \in \mathbb{R}^3$ such that $\dot{R}_1^0 = S(w_1^0)R_1^0$
  - This means if we can find w_1^0 we can compute the derivative of a rotation matrix easily without taking the derivative for each term in the matrix
  - Proof: differentiate $R_1^0(R_1^0)^T = I$
    * $\dot{R}_1^0(R_1^0)^T + R_1^0(\dot{R}_1^0)^T = 0$
    * $\dot{R}_1^0(R_1^0)^T = -R_1^0(\dot{R}_1^0)^T = -\left(\dot{R}_1^0(R_1^0)^T\right)^T$
    * This means that $\dot{R}_1^0(R_1^0)^T$ is skew-symmetric, so there exists w_1^0 such that $\dot{R}_1^0(R_1^0)^T = S(w_1^0)$
    * Rearrange to yield $\dot{R}_1^0 = S(w_1^0)R_1^0$
- Consider a point moving in frame 0, $p^0(t)$; its linear velocity is defined as $v^0(t) = \frac{\mathrm{d}}{\mathrm{d}t}p^0(t) = \dot{p}^0(t)$
- Now suppose frame 1 is moving relative to frame 0, i.e. O_1^0, R_1^0 are functions of t; consider a point p fixed in frame 1
  - $p^0 = O_1^0 + R_1^0 p^1$
  - Differentiate: $$\begin{aligned} \dot{p}^0 &= \dot{O}_1^0 + \dot{R}_1^0 p^1 + R_1^0 \dot{p}^1 \\ &= \dot{O}_1^0 + S(w_1^0)R_1^0 p^1 + R_1^0 \dot{p}^1 \\ &= \dot{O}_1^0 + w_1^0 \times (R_1^0 p^1) + R_1^0 \dot{p}^1 \end{aligned}$$
  - $\dot{O}_1^0$ is the *linear velocity* of frame 1 with respect to frame 0
  - w_1^0 is the *angular velocity* of frame 1 with respect to frame 0
- Consider frames 0, 1, 2, where 0 is the inertial frame and frames 1 and 2 are moving; how can we express w_2^0 in terms of w_1^0 and w_2^1?
  - We know $\dot{R}_2^0 = S(w_2^0)R_2^0$ and $R_2^0 = R_1^0 R_2^1$
  - Take derivative: $$\begin{aligned} \dot{R}_2^0 &= \dot{R}_1^0 R_2^1 + R_1^0 \dot{R}_2^1 \\ &= S(w_1^0)R_1^0 R_2^1 + R_1^0 S(w_2^1)R_2^1 \\ &= S(w_1^0)R_2^0 + R_1^0 S(w_2^1)(R_1^0)^T R_1^0 R_2^1 \\ &= S(w_1^0)R_2^0 + S(R_1^0 w_2^1)R_1^0 R_2^1 \\ &= S(w_1^0)R_2^0 + S(R_1^0 w_2^1)R_2^0 \\ &= (S(w_1^0) + S(R_1^0 w_2^1))R_2^0 \\ &= S(w_1^0 + R_1^0 w_2^1)R_2^0 \end{aligned}$$
  - Compare this with the above, we get $S(w_2^0) = S(w_1^0 + R_1^0 w_2^1) \implies w_2^0 = w_1^0 + R_1^0 w_2^1$
- Note the similarity between $w_2^0 = w_1^0 + R_1^0 w_2^1$ and the transformation for points, $O_2^0 = O_1^0 + R_1^0 O_2^1$
  - Angular velocities work similarly to points – we can “add” them
- Therefore, for an n-link manipulator:
  - $O_i^0 = O_{i-1}^0 + R_{i-1}^0 O_i^{i-1}$
  - $\dot{O}_i^0 = \dot{O}_{i-1}^0 + \dot{R}_{i-1}^0 O_i^{i-1} + R_{i-1}^0 \dot{O}_i^{i-1}$

  - $w_i^0 = w_{i-1}^0 + R_{i-1}^0 w_i^{i-1}$
  - This is a set of recursive formulas for computing the end-effector linear and angular velocities
- Next, we can write explicit formulas for w_i^{i-1} and $\dot{O}_i^{i-1}$, which will be easy to do since all joints operate in the z axis