Tutorial 5, Feb 16, 2024

Vector Random Variables and Joint Gaussians

- Let $\boldsymbol{X} = (X_1, \dots, X_n)^T \in \mathbb{R}^n$ represented by a column vector
- The expectation is $\boldsymbol{\mu} = E[\boldsymbol{X}] = (E[X_1], \dots, E[X_n])^T$
- The covariance matrix is $\boldsymbol{\Sigma} = E[(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^T]$
  - $\boldsymbol{u}^T \boldsymbol{\Sigma} \boldsymbol{u} = \boldsymbol{u}^T E[(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^T]\boldsymbol{u} = E[\boldsymbol{u}^T(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^T\boldsymbol{u}] = E[(\boldsymbol{u}^T(\boldsymbol{X} - \boldsymbol{\mu}))^2] > 0$
  - Therefore $\boldsymbol{\Sigma}$ is symmetric positive definite
- Consider a linear transformation $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}$
  - $\boldsymbol{\mu}_{\boldsymbol{Y}} = \boldsymbol{A}\boldsymbol{\mu}_{\boldsymbol{X}}$
  - $\boldsymbol{\Sigma}_{\boldsymbol{Y}} = \boldsymbol{A}\boldsymbol{\Sigma}_{\boldsymbol{X}}\boldsymbol{A}^T$
- $\boldsymbol{X}$ is jointly Gaussian if $f_{\boldsymbol{X}}(\boldsymbol{x}) = \dfrac{1}{(2\pi)^{\frac{n}{2}} \det(\boldsymbol{\Sigma}_{\boldsymbol{X}})^{\frac{1}{2}}} \exp\left[-\dfrac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu}_{\boldsymbol{X}})^T \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1} (\boldsymbol{x} - \boldsymbol{\mu}_{\boldsymbol{X}})\right]$
- For any linear transformation $\boldsymbol{A}$, if $\boldsymbol{X}$ is jointly Gaussian then $\boldsymbol{A}\boldsymbol{X}$ is also jointly Gaussian
  - This is because the sum of Gaussians is also Gaussian
- If $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}_{\boldsymbol{X}}, \boldsymbol{\Sigma}_{\boldsymbol{X}})$ and $\boldsymbol{\Sigma}_{\boldsymbol{X}} = \boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^T$, then $\boldsymbol{Z} = \boldsymbol{Q}^T\boldsymbol{X}$ would have a diagonal covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{Z}} = \boldsymbol{D}$, i.e. it will consist of all independent Gaussians
  - Using this we can find a transformation of the original variables that makes them independent