Tutorial 4, Feb 9, 2024

Hypothesis Testing

Definition

Hypothesis Testing: Let X be an RV with distribution $f_X(x;H_0)$ or $f_X(x;H_1)$, and we observe x; we would like to infer whether H_0 or H_1 is true by designing a decision rule $g(x)$.

- Generally we partition S into regions A_0, A_1, so $g(x) = \begin{cases} H_0 & x \in A_0 \\ H_1 & x \in A_1 \end{cases}$
- A good decision rule minimizes both the probability of type I errors α (false rejection of H_0/false acceptance of H_1) and type II errors β (false acceptance of H_0/false rejection of H_1)
  - $\alpha = P[X \notin A_0; H_0] = \int_{A_0^c} f(x;H_0)\,\mathrm{d}x$
  - $\beta = P[X \in A_0; H_1] = \int_{A_0} f(x;H_1)\,\mathrm{d}x$

Definition

Neyman-Pearson Lemma: For a given target α, the minimum β is achieved by a decision rule of the form

$$\frac{f(x;H_0)}{f(x;H_1)} \underset{H_1}{\overset{H_0}{\gtrless}} \zeta$$

- Setting ζ to 1 gives MLE, but in general we can set this to anything
- If we have priors $\pi_0 = P[H_0], \pi_1 = P[H_1]$ then $\frac{\pi_0 f(x;H_0)}{\pi_1 f(x;H_1)} \underset{H_1}{\overset{H_0}{\gtrless}} \zeta$
- Example: Let X be the number of coin tosses until the first heads
  - H_0: fair coin with $P[H] = \frac{1}{2}$
  - H_1: biased coin with $P[H] = \frac{1}{8}$
  - The number of tosses until first heads is given by a geometric distribution
  - With ML:
    * $p(x;H_0) = \left(\frac{1}{2}\right)^x$
    * $p(x;H_1) = \left(1 - \frac{1}{8}\right)^{x-1}\left(\frac{1}{8}\right)$
    * Test: $\left(\frac{1}{2}\right)^x \underset{H_1}{\overset{H_0}{\gtrless}} \left(1 - \frac{1}{8}\right)^{x-1}\left(\frac{1}{8}\right)$
    * Log both sides: $-x \underset{H_1}{\overset{H_0}{\gtrless}} (x-1)\log\left(1 - \frac{1}{8}\right) + \log\left(\frac{1}{8}\right)$
    * $x \underset{H_0}{\overset{H_1}{\gtrless}} \frac{3 + \log(7/8)}{1 + \log(7/8)} \approx 3.5$
    * Type I error: $\alpha = P[X \geq 4; H_0] = \sum_{x=4}^{\infty} \left(\frac{1}{2}\right)^x = \frac{1}{8}$
    * Type I error: $\beta = P[X \leq 3; H_1] = \sum_{x=1}^{3} \left(1 - \frac{1}{8}\right)^{x-1}\left(\frac{1}{8}\right)$
- Example: $f(x;H_0)$ is uniform on $[0,1]$; $f(x;H_1)$ is uniform on $[-a,a]$
  - With ML:
    * If $a < \frac{1}{2}$ then select H_1 if $x \in [-a,a]$ and H_0 otherwise

- This is because with $a < \frac{1}{2}$ the likelihood of H_1 is always greater, as long as x falls within the interval

* If $a \geq \frac{1}{2}$ then select H_0 if $x \in [0, 1]$ and H_1 otherwise
  - Again with $a \geq \frac{1}{2}$ the likelihood of H_1 is always less, as long as x falls within the interval of H_0
* Note this assumes we never see an x outside these distributions