Midterm Review, Mar 1, 2024

2019 Midterm Q2

- Suppose that $x_1, \ldots, x_n$ are IID realizations of a uniform RV on $[\theta - 1, \theta + 1]$, where θ is an unknown parameter.

Part (a)

- Suppose that $n = 6$ and $(x_1, x_2, x_3, x_4, x_5, x_6) = (5.5, 6.1, 6.8, 5.2, 6.0, 5.7)$; what is the MLE of θ given $x_1, \ldots, x_6$?
- $f(x|\theta) = \begin{cases} \dfrac{1}{2} & x \in [\theta - 1, \theta + 1] \\ 0 & \text{otherwise} \end{cases}$
- $f(x_1, \ldots, x_6|\theta) = f(x_1|\theta) \ldots f(x_n|\theta) = \begin{cases} \dfrac{1}{2^6} & \theta - 1 < \theta_i < \theta + 1, i = 1, \ldots, 6 \\ 0 & \text{otherwise} \end{cases}$
- This means $\max x_i - 1 < \theta$ and $\min x_i + 1 > \theta$ for the likelihood to be nonzero, but it is constant otherwise
- Therefore any θ in the range $[5.8, 6.2]$ is valid

Part (b)

- Suppose that now θ has a prior distribution that is uniform on $[0, 6]$; what is the MAP estimate of θ?
- $f(\theta) = \begin{cases} 1/6 & 0 < \theta < 6 \\ 0 & \text{otherwise} \end{cases}$
- $f(\boldsymbol{x}|\theta)f(\theta) = \begin{cases} \dfrac{1}{2^6 \cdot 6} & \max x_i - 1 < \theta < \min x_i \text{ and } 0 < \theta < 6 \\ 0 & \text{otherwise} \end{cases}$
- Since now we impose $0 < \theta < 6$, the MAP estimate is now any value in $[5.8, 6.0]$

Part (c)

- Given the data and prior in the previous parts, what is the LMS estimate?
- $\hat{\theta}_{\text{LMS}} = E[\Theta|\boldsymbol{x}] = \int \theta f(\theta|\boldsymbol{x})\, \mathrm{d}\theta$
- $f(\theta|\boldsymbol{x}) = \dfrac{f(\boldsymbol{x}|\theta)f(\theta)}{f(x)} = \begin{cases} \dfrac{c}{2^6 \cdot 6} & 5.8 < \theta < 6 \\ 0 & \text{otherwise} \end{cases}$
- If we carry out the integral we find that $\hat{\theta}_{\text{LMS}} = 5.9$