Lecture 16, Mar 15, 2024

Convergence of Markov Chains

- We would like to know the following:
  1. When does a Markov chain have steady-state probabilities?
  2. Are the steady-state probabilities unique?
  3. When does P^n converge to the steady-state probabilities?
- Suppose that state i is recurrent; let $T_i(k)$ be the number of steps between the kth visit and $k-1$th visit
  - The proportion of time spent in i is $\dfrac{k}{T_i(1) + \cdots + T_i(k)}$
  - Note that this is the reciprocal of $\dfrac{1}{k}\sum_{j=1}^{k} T_i(j) \to E[T_i]$
    * This is because the return times T_i are IID
  - Therefore this converges to $\dfrac{1}{E[T_i]} = \pi_i$
    * This satisfies the global balance equations
- State i is *positive recurrent* if $E[T_i] < \infty$; in this case all states in the class will have nonzero probability since $\pi_i > 0$
  - A positive recurrent, aperiodic state is *ergodic*
- State i is *null recurrent* if $E[T_i] = \infty$; in this case $\pi_i = 0$ and states will have zero probability
  - This arises with infinite state Markov chains
  - Even though we always come back (the state is recurrent), if the chain is infinite, it is possible to have zero probability if we don't return often enough

Theorem

For an irreducible (single class), aperiodic, and positive recurrent Markov chain,

$$\lim_{n\to\infty} p_{jj}(n) = \pi_j \quad \forall j$$

that is, there is a steady-state PMF, obtainable by solving the global balance equations.

Theorem

For an irreducible, periodic, and positive recurrent Markov chain with period d,

$$\lim_{n\to\infty} p_{jj}(nd) = d\pi_j \quad \forall j$$

where the "steady-state" probabilities exist over the period instead of individual steps.

- The factor of d comes in because there are only $\dfrac{1}{d}$ of the total states that can occur
- For a periodic Markov chain the transition probabilities form multiple sub-matrices