Lecture 15, Mar 11, 2024

Discrete-Time Markov Chains

- A *Markov chain* is a discrete-valued random sequence X_n where the future is of the process given the present is independent of the past, i.e. $P[X_{n+1}|X_1, \dots, X_n] = P[X_{n+1}|X_n]$
  - The present X_n is known as the *state*
- Example: sum process $S_n = X_1 + \cdots + X_n = S_{n-1} + X_n$ where X_i are IID, $S_0 = 0$
  - $P[S_{n+1} = s_{n+1}|S_n = s_n, \dots, S_1 = s_1] = P[X_n = s_{n+1} - s_n] = P[S_{n+1} = s_{n+1}|S_n = s_n]$
- $P[X_3, X_2, X_1] = P[X_3|X_2]P[X_2|X_1]P[X_1]$ due to the Markov property
  - The latter is a lot easier to store since we don't have to go over all possible comminations of the 3 variables
  - $P[X_3|X_2]$ and $P[X_2|X_1]$ are *transition probabilities*
  - $P[X_1]$ is the *initial probability*
  - The joint PMF of the values is the product of the initial probability and all intermediate transition probabilities
- These transition probabilities could be time-dependent, but often they are constant, in which case the Markov chain has *homogeneous transition probabilities*
  - We only need to store a single version of the transition probability matrix
- X_n is completely specified by $p_i(0)$ and the *transition probability matrix*, where the entry ij denotes the probability of transitioning to state j while in state i
  - Each row sums up to 1 since it s a PMF
  - We can also use a *state transition diagram* to visualize this
- Example: speech activity using a Markov model; if packet n was silent, then the probability of silence in the next packet is $1 - \alpha$ and probability of speech activity is α
  - Transition matrix: $P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$

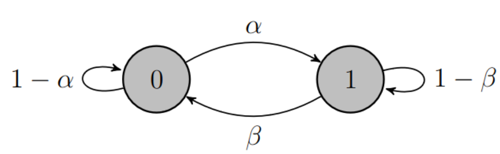

Figure 1: State transition diagram for example.

- The transition probability matrix after n steps, $P(n)$, is the original single-step transition matrix raised to the power n; $p_{ij}(n)$ denotes the probability of transition for n steps
  - $P(n) = P(n-1)P = P^n$
  - Suppose we start with some initial state PMF $\boldsymbol{p}(0)$, then after n steps the new distribution will be $\boldsymbol{p}(n) = \boldsymbol{p}(0)P^n$
  - To find a closed-form expression we can diagonalize P
- Some Markov chains will have $\lim_{n\to\infty} P(n)$ exist; this will give the asymptotic or steady-state PMF
- Instead of finding an expression for P^n, we can find the steady-state PMF more directly, assuming one exists
  - Let the steady-state PMF be $\pi = (\pi_1, \dots, \pi_n)$ and so $p_{ij}(n) = \pi_j$
  - To find the steady-state PMF we solve $\boldsymbol{\pi} = \boldsymbol{\pi}P$ such that $\sum_i \pi_i = 1$
    - Since $\boldsymbol{\pi}$ is a PMF, the first equation only gives $n-1$ independent equations
    - i.e. when we have this PMF, it remains unchanged by the Markov chain
  - These together are known as the *global balance equations*

Recurrence Relations in Markov Chains

- When does a Markov chain have steady-state probabilities?

- We can break states into separate *classes*, where each one is of a different type
  - State j is *accessible* from i if there is a sequence of transitions from i to j with nonzero probability
  - States i and j *communicate* if they are accessible from each other (i.e. we can go from i to j and back); this is denoted $i \leftrightarrow j$
    * A state always communicates with itself by definition (even if there are no self edges)
    * $i \leftrightarrow j, j \leftrightarrow k \implies i \leftrightarrow k$
  - States that communicate with each other are in the same *class*
    * States in the same class share the same fate – they have the same limiting behaviour
  - Classes are always disjoint (i.e. one state cannot be in two different classes; in that case the two classes would communicate with each other so they would be the same)
    * However, states in different classes aren't necessarily independent, since we can still have one-way accessible connections
- The states of a Markov chain consists of one or more disjoint classes; if it has a single class, it is *irreducible*
  - Intuitively this means that we can go from one state to any other state
- A state is *periodic* with period d if it can only re-occur at times that are multiples of d, i.e. $p_{ii}(n) = 0$ if n is not a multiple of d
  - If the Markov chain is periodic, it's similar to having multiple chains
  - Let $\tau(x)$ be all the possible times that we can visit x (starting from x at time 0); the period of x is the GCD of $\tau(x)$ (same across an entire class)
  - If all states/classes have period 1, the chain is *aperiodic*
- A Markov chain that is irreducible and aperiodic will converge to a stationary distribution (we see this later)

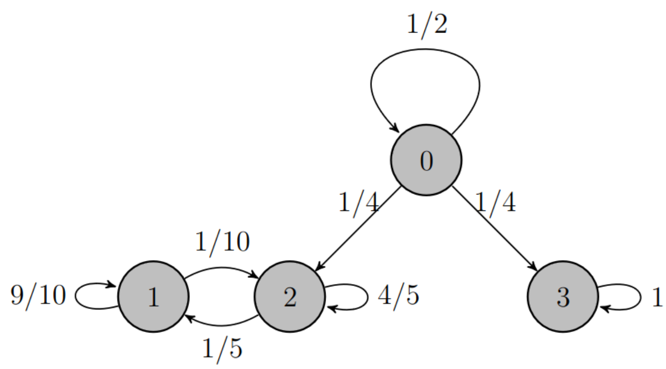

Figure 2: State transition diagram for a Markov chain with 3 classes: $\{0\}, \{1,2\}, \{3\}$. This is aperiodic.

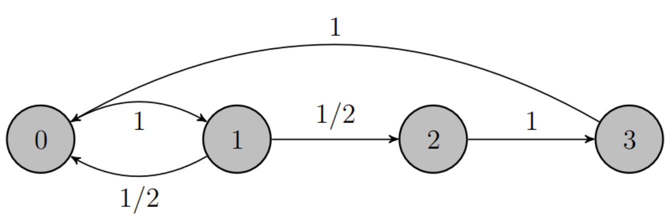

Figure 3: State transition diagram for a Markov chain with a single class. This is periodic with period 2.

- A state i is *recurrent* if the process returns to the state with probability 1, or *transient* if the probability is less than 1
  - For any recurrent state, if we leave it, we know eventually we will come back
  - Whenever we come back we will go through the exact same cycle again
  - For a transient state, when we leave it, it's possible that we won't reach this state again

- Recurrence/transience is a class property; if a state is recurrent then all states in its class will also be recurrent
- To find if a state is recurrent, we sum the probability of returning to the state after all possible number of steps
  - State i is recurrent iff $\sum_{n=1}^{\infty} p_{ii}(n) = \infty$
  - State i is transient iff $\sum_{n=1}^{\infty} p_{ii}(n) < \infty$
  - e.g. if $p_{00}(n) = \left(\frac{1}{2}\right)^n$, then the sum converges to 2 so it is finite and the state is transient

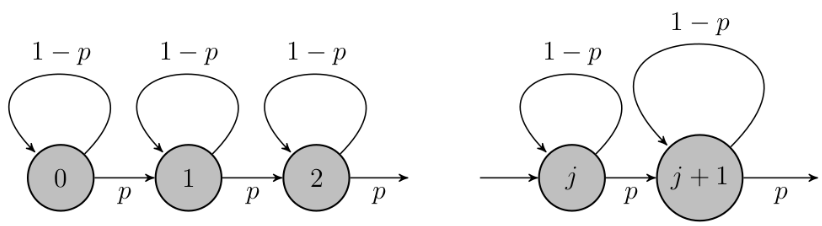

Figure 4: Binomial counting process state transition diagram. Each state is its own class and every class/state is transient.

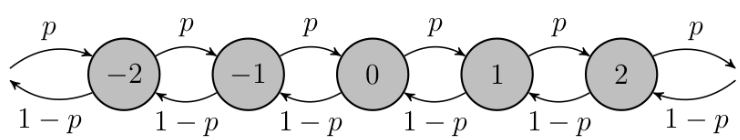

Figure 5: Random walk process state transition diagram. All states are in the same class. This is also periodic.