Lecture 12, Feb 16, 2024

Gaussian Systems

- Let $\boldsymbol{X}$ be jointly Gaussian and let $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{b} + \boldsymbol{Z}$, where $\boldsymbol{Z} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma_Z})$
  - Assume that $\boldsymbol{A}, \boldsymbol{b}$ are fixed and known, and $\boldsymbol{Z}, \boldsymbol{X}$ are independent (zero-mean, independent noise)
  - We would like to estimate $\boldsymbol{X}$ from $\boldsymbol{Y}$
- Again let $\boldsymbol{W} = \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{A}\boldsymbol{X} + \boldsymbol{b} + \boldsymbol{Z} \end{bmatrix} = \begin{bmatrix} \boldsymbol{1} & \boldsymbol{0} \\ \boldsymbol{A} & \boldsymbol{1} \end{bmatrix} \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Z} \end{bmatrix} + \begin{bmatrix} 0 \\ \boldsymbol{b} \end{bmatrix}$
  - Since $\boldsymbol{W}$ is obtained through a linear transformation from $\begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Z} \end{bmatrix}$, we know it is jointly Gaussian
  - We've converted this to the conditional PDF problem we found last time
  - $\hat{\boldsymbol{x}}_{\text{MAP/LMS}}(\boldsymbol{y}) = \boldsymbol{\mu_X} + \boldsymbol{\Sigma_{XY}}\boldsymbol{\Sigma_{YY}^{-1}}(\boldsymbol{y} - \boldsymbol{\mu_Y})$
- $\boldsymbol{\mu_Y} = E[\boldsymbol{A}\boldsymbol{X} + \boldsymbol{b} + \boldsymbol{Z}] = \boldsymbol{A}\boldsymbol{\mu_X} + \boldsymbol{b}$
- $\boldsymbol{\Sigma_{XY}} = E[(\boldsymbol{X} - \boldsymbol{\mu_X})(\boldsymbol{A}\boldsymbol{X} + \boldsymbol{b} + \boldsymbol{Z} - \boldsymbol{A}\boldsymbol{\mu_X} - \boldsymbol{b})^T]$

  $= E[(\boldsymbol{X} - \boldsymbol{\mu_X})(\boldsymbol{X} - \boldsymbol{\mu_X})^T \boldsymbol{A}^T] + E[(\boldsymbol{X} - \boldsymbol{\mu_X}) + \boldsymbol{Z}]$

  $= \boldsymbol{\Sigma_X}\boldsymbol{A}^T$
- $\boldsymbol{\Sigma_{YY}} = E[(\boldsymbol{A}(\boldsymbol{X} - \boldsymbol{\mu_X}) + \boldsymbol{Z})(\boldsymbol{A}(\boldsymbol{X} - \boldsymbol{\mu_X}) + \boldsymbol{Z})^T]$

  $= \boldsymbol{A}\boldsymbol{\Sigma_X}\boldsymbol{A}^T + \boldsymbol{\Sigma_Z}$
- Substituting these in we get $\hat{\boldsymbol{x}}_{\text{MAP/LMS}} = \boldsymbol{\mu_X} + \boldsymbol{\Sigma_X}\boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{\Sigma_X}\boldsymbol{A}^T + \boldsymbol{\Sigma_Z})^{-1}(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{\mu_X} - \boldsymbol{b})$

  $= (\boldsymbol{\Sigma_X^{-1}} + \boldsymbol{A}^T\boldsymbol{\Sigma_Z^{-1}}\boldsymbol{A})^{-1}(\boldsymbol{A}^T\boldsymbol{\Sigma_Z^{-1}}(\boldsymbol{y} - \boldsymbol{b}) + \boldsymbol{\Sigma_X^{-1}}\boldsymbol{\mu_X})$
  - Note the second form can be derived using the matrix inversion formula
    - It only uses the inverse covariance (precision) matrices
  - Also $\boldsymbol{\Sigma}_{X|y} = (\boldsymbol{\Sigma_X^{-1}} + \boldsymbol{A}^T\boldsymbol{\Sigma_Z^{-1}}\boldsymbol{A})^{-1}$
- Example: consider $\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \Theta + \begin{bmatrix} W_1 \\ \vdots \\ W_n \end{bmatrix} \iff \boldsymbol{Y} = \boldsymbol{A}X + \boldsymbol{Z}$
  - Let $\theta \sim \mathcal{N}(\boldsymbol{x}_0, \sigma_0^2)$ and $w_i \sim \mathcal{N}(0, \sigma_i^2)$
  - θ is some true value, plus zero-mean Gaussian noise w_i; we measure this n times
  - Compute terms:
    - $\boldsymbol{b} = \boldsymbol{0}$
    - $\boldsymbol{\mu_X} = x_0$
    - $\boldsymbol{\mu_Y} = \boldsymbol{A}\boldsymbol{\mu_X} + \boldsymbol{b} = \begin{bmatrix} x_0 \\ \vdots \\ x_0 \end{bmatrix}$
    - $\boldsymbol{\Sigma_{YY}} = \boldsymbol{A}\boldsymbol{\Sigma_X}\boldsymbol{A}^T + \boldsymbol{\Sigma_Z} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \sigma_0^2 \begin{bmatrix} 1 & \dots & 1 \end{bmatrix} + \begin{bmatrix} \sigma_1^2 & & \\ & \vdots & \\ & & \sigma_n^2 \end{bmatrix}$
    - $\boldsymbol{\Sigma_{XY}} = \boldsymbol{\Sigma_X}\boldsymbol{A}^T = \begin{bmatrix} \sigma_0^2 & \dots & \sigma_0^2 \end{bmatrix}$
  - Substituting, we get $\dfrac{\frac{x_0}{\sigma_0^2} + \sum_{i=1}^n \frac{x_i}{\sigma_i^2}}{\frac{1}{\sigma_0^2} + \sum_{i=1}^n \frac{1}{\sigma_i^2}}$
    - This is the same result that we would get through regular MAP estimation