Lecture 10, Feb 9, 2024

Gaussian Random Vectors

Definition

Gaussian Random Vector: $\boldsymbol{X} \in \mathbb{R}^n$ is Gaussian distributed with mean $\boldsymbol{m_X}$ and covariance $\boldsymbol{K_X}$ if it has distribution

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{n}{2}} \det(\boldsymbol{K_X})^{\frac{1}{2}}} \exp\left[-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{m_X})^T \boldsymbol{K_X}^{-1}(\boldsymbol{x} - \boldsymbol{m_X})\right]$$

- The exponent is in quadratic form and specifies an ellipsoid in $\mathbb{R}^n$
- Note that if $X_1, \dots, X_n$ are all uncorrelated then $\boldsymbol{K_X}$ is diagonal
  - $\boldsymbol{K_X} = \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix}$
  - $\boldsymbol{K_X}^{-1} = \begin{bmatrix} \frac{1}{\sigma_1^2} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n^2} \end{bmatrix}$
  - Multiply this by $\boldsymbol{x} - \boldsymbol{m_X}$ and we get $\left(\frac{x_1 - m_1}{\sigma_1}\right)^2 + \cdots + \left(\frac{x_n - m_n}{\sigma_n}\right)^2$
  - This expression is in the exponent, so we can split it up into a product of exponentials
  - The resulting distribution is a product of distributions in each X, so they are all independent
- Consider some linear transformation $\boldsymbol{A}$ so that $\boldsymbol{Y} = \boldsymbol{AX}$ is the transformed version of $\boldsymbol{X}$, which are jointly Gaussian
  - $f_{\boldsymbol{Y}}(\boldsymbol{y}) = \frac{f_{\boldsymbol{X}}(\boldsymbol{x})}{\det \boldsymbol{A}} = \frac{f_{\boldsymbol{X}}(\boldsymbol{A}^{-1}\boldsymbol{y})}{\det \boldsymbol{A}}$
  - Substitute this into the Gaussian for $\boldsymbol{X}$, in the exponent we get $(\boldsymbol{A}^{-1}\boldsymbol{y} - \boldsymbol{m_X})^T \boldsymbol{K_X}^{-1}(\boldsymbol{A}^{-1}\boldsymbol{y} - \boldsymbol{m_X})$
  - Factor out $\boldsymbol{A}$: $(\boldsymbol{y} - \boldsymbol{Am_X})^T \boldsymbol{A}^{-T} \boldsymbol{K_X}^{-1} \boldsymbol{A}^{-1}(\boldsymbol{y} - \boldsymbol{Am_X}) = (\boldsymbol{y} - \boldsymbol{Am_X})^T (\boldsymbol{AK_XA}^T)^{-1}(\boldsymbol{y} - \boldsymbol{Am_X})$
  - Therefore $\boldsymbol{AK_XA}^T$ is the new covariance matrix and $\boldsymbol{Am_X}$ is the new mean; the result is still Gaussian
  - Since $\boldsymbol{K_X}$ is real and symmetric we can find $\boldsymbol{A}$ such that $\boldsymbol{AK_XA}^T = \boldsymbol{\Lambda}$, then the resulting Gaussian will be independent in its variables
- Suppose $\boldsymbol{X}$ is IID, can we find a linear transformation $\boldsymbol{A}$ such that the resulting $\boldsymbol{Y} = \boldsymbol{AX}$ has covariance $\boldsymbol{K_Y}$?
  - $\boldsymbol{K_Y} = \boldsymbol{P\Lambda P}^T = \boldsymbol{P\Lambda}^{\frac{1}{2}}\boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{P}^T$
  - Let $\boldsymbol{A} = \boldsymbol{P\Lambda}^{\frac{1}{2}}$
  - Then $\boldsymbol{K_Y} = \boldsymbol{AK_XA}^T = \boldsymbol{A1A}^T = \boldsymbol{AA}^T = \boldsymbol{P\Lambda}^{\frac{1}{2}}\boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{P}^T = \boldsymbol{P\Lambda P}^T$