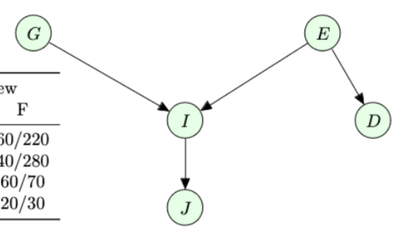
Lecture 7, Feb 27, 2024

Bayesian Networks

- So far we have only discussed deterministic, fully observable task environments
- Partially observable or stochastic environments can be modelled with probability
- Often we have multiple models of our state, and then based on evidence, we classify which model is correct (or which one we're most likely to be in)
- How can we store conditional probabilities efficiently?
  - If we have n variables each taking 2 values, to store the conditional probability over all combinations of variables we'd need 2^n entries
  - Not all variables may be dependent on each other; how can we take advantage of this?
- A *Bayesian network* is a probabilistic graphical model representing a set of variables and their conditional dependence via a directed acyclic graph (DAG)
  - In the DAG, an edge $A \to B$ denotes that B is conditionally dependent on A
  - Traversing the DAG gives us a chain of dependence between events
  - Often human intuition is used to determine which events have a causal relationship
  - At each node, we store a conditional probability table for the probability of the event at the node, given all its parents
    - * The table has an entry for every combination of its parents' values
  - If a node has no parents, we simply store the absolute probability of that event (not conditioned on anything)
- We don't need to collect data about every possible event from the same sample, i.e. we may compute probabilities separately, using different datasets, for different nodes
  - However we always assume that whatever sample we take is representative of the population
  - This means we can combine different studies
- Bayesian networks allow compact representation of probability distributions
  - For a network over n nodes, if a node has at max q parents, then the space complexity is $O(n \cdot 2^q)$, which is often significantly less than 2^n

Grades	
T	F
500/600	100/600

ESP	
T	F
290/600	310/600



Grades	ESP	Interview T	Interview F
T	T	160/220	60/220
T	F	240/280	40/280
F	T	10/70	60/70
F	F	10/30	20/30

ESP	Driver T	Driver F
T	230/300	70/300
F	120/300	180/300

Interview	Job T	Job F
T	380/450	70/450
F	5/100	95/100

Figure 1: Example Bayesian network.

- The crucial assumption of Bayesian networks is the *Bayesian Network Law*: for any node, given its parents, its probability is completely independent of its non-descendants; i.e. nothing that came before it matters except for its parents
  - Note: v is a descendant of u if there is a directed path from u to v
  - Note that the probability of a node can still depend on its descendants
  - This also works even if we don't give the direct parents, as long as the probability of all the direct parents can be computed from the grandparents given

- e.g. in the above graph, $P(I|G, E, D) = P(I|G, E)$ since I is not a descendant of D
- Example: in the graph below:
  - A and E are not independent
  - A and E, given B, are independent
  - A and E, given G, C, are independent
  - A and E, given G only, are not independent

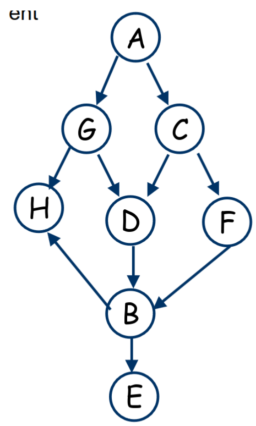


Figure 2: Example DAG network.

- For any set of events, $P(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} P(x_i|x_{i+1}, \ldots, x_n)$

$$= P(x_1|x_2, \ldots, x_n)P(x_2|x_3, \ldots, x_n) \ldots P(x_{n-1}|x_n)P(x_n)$$

  - Given the Bayesian network law, we can simplify these terms significantly by taking out all the variables except for the direct parents
  - e.g. $P(G, I, J, E, D) = P(J, I, G, D, E)$

$$= P(J|I, G, D, E)P(I|G, D, E)P(G|D, E)P(D|E)P(E)$$
$$= P(J|I)P(I|G, E)P(G)P(D|E)P(E)$$

  - This means that we can always compute $P(x_1, \ldots, x_n)$ just by looking at the conditional probabilities stored in the network
    * There are multiple ways to expand this joint probability, but there will always be one order that works
    * The order that works is determined by the topological sorting of the graph
  - To compute the joint probability over all the events in the network, compute the product of each event conditioned on its immediate parents
    * Always guaranteed to work due to the existence of a topological sort as above
- Using Bayesian networks, we can compute the probability of 2^n events using only $n \cdot 2^q$ entries