AER372 Control Systems Reference Sheet

Dynamic System Response

- The response of an LTI system to $u(t)$ is $y(t) = \int_0^t u(\tau)h(t-\tau)\,\mathrm{d}\tau = u(t) * h(t)$ where $h(t)$ is the response of the system to the unit impulse $\delta(t)$; convolution has the properties:
  1. Commutativity: $x_1(t) * x_2(t) = x_2(t) * x_1(t)$
  2. Associativity: $x_1(t) * [x_2(t) * x_3(t)] = [x_2(t) * x_2(t)] * x_3(t)$
  3. Distributivity: $x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_2(t) * x_3(t)$
  4. Shift: $x_1(t) * x_2(t-T) = x_1(t-T) * x_2(t)$
  5. Impulse: $x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau) = x(t)$
  6. Width: the convolution of a function covering a length of time T_1 and another function covering T_2 covers a time of $T_1 + T_2$
- Laplace transform: $F(s) = \mathcal{L}\{f(t)\} \equiv \int_{0^-}^{\infty} f(t)e^{-st}\,\mathrm{d}t$ (see table of transforms)

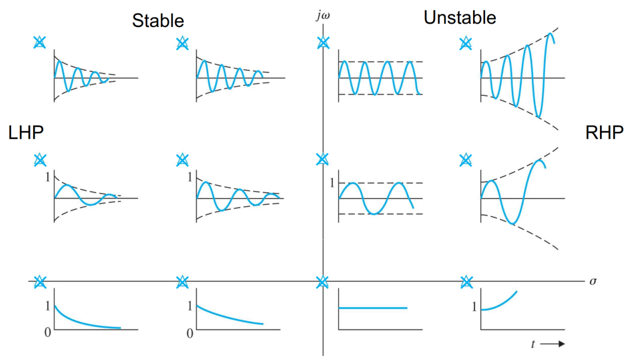

Figure 1: Behaviour of a system according to its poles.

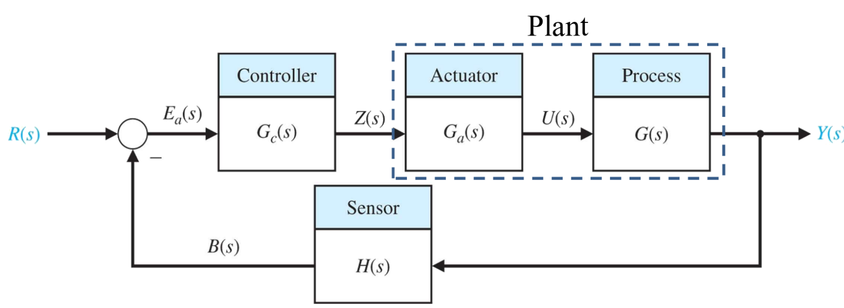

Figure 2: General feedback system.

- Transfer function definitions for a general feedback system:
  - Closed-loop: $\mathcal{T}(s) = \dfrac{Y(s)}{R(s)} = \dfrac{G_c(s)G_a(s)G(s)}{1 + G_c(s)G_a(s)G(s)H(s)}$
  - Open-loop: $L(s) = \dfrac{B(s)}{E_a(s)} = G_c(s)G_a(s)G(s)H(s)$
  - Error: $\dfrac{E(s)}{R(s)} = \dfrac{R(s) - Y(s)}{R(s)} = \dfrac{1 + G_c(s)G_a(s)G(s)(H(s) - 1)}{1 + G_c(s)G_a(s)G(s)H(s)}$
    * Note the $E(s)$ here is not the same as $E_a(s)$
  - Feedforward: $\dfrac{Y(s)}{E_a(s)} = G_c(s)G_a(s)G(s)$

- Feedback: $\dfrac{B(s)}{R(s)} = \dfrac{G_c(s)G_a(s)G(s)H(s)}{1 + G_c(s)G_a(s)G(s)H(s)}$
- Sensitivity: $\mathcal{S}(s) = \dfrac{1}{1 + G_c(s)G_a(s)G(s)H(s)}$

- Stability:
  - Bounded-Input-Bounded-Output (BIBO): any bounded input creates bounded output (no convergence requirement)
  - Asymptotic: any initial condition decays to 0
  - Marginal/neural: (for zero input) any initial condition generates a bounded output
- Routh array: only LHP roots if all elements in the first column are positive; number of RHP roots is equal to the number of sign changes

First-Order System

- $H(s) = \dfrac{b}{s+a} \implies h(t) = be^{-at} \implies y_s(t) = \dfrac{b}{a}\left(1 - e^{-at}\right)$
- DC gain: $\dfrac{b}{a}$
- Time constant: $T = \dfrac{1}{a}$
- Rise time: $t_r \approx 2.2T$ (10% to 90%)
- Settling time: $t_s \approx \dfrac{4.6}{a}$ (0 to 99%)

Second-Order System

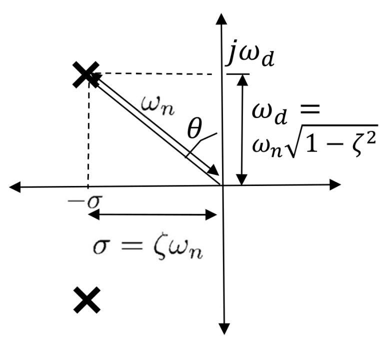

Figure 3: Pole locations for underdamped system.

- $H(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$; poles: $-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$
  - Overdamped: $\zeta > 1$, then $-\sigma = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \iff \omega_n = \sqrt{\sigma_1\sigma_2}, \zeta = \dfrac{\sigma_1 + \sigma_2}{2\sqrt{\sigma_1 + \sigma_2}}$
  - Critically damped: $\zeta = 1$, then $\sigma = \omega_n$
  - Underdamped: $\zeta < 1$, then $\sigma = \zeta\omega_n, \omega_d = \omega_n\sqrt{1 - \zeta^2}$, poles at $s = -\sigma \pm j\omega_d$
    - $H(s) = \dfrac{\sigma^2 + \omega_d^2}{(s+\sigma)^2 + \omega_d^2}$
  - **Analysis below assumes underdamped case**
- $h(t) = \dfrac{\sigma^2 + \omega_d^2}{\omega_d} e^{-\sigma t}\sin(\omega_d t) = \dfrac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)$
- $y_s(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \dfrac{\sigma}{\omega_d}\sin(\omega_d t)\right) = 1 - e^{-\sigma t}\dfrac{\omega_n}{\omega_d}\cos(\omega_d t - \phi)$
  - $\phi = \tan^{-1}\left(\dfrac{\omega_d}{\sigma}\right) = \tan^{-1}\left(\dfrac{\omega_d}{\zeta\omega_n}\right)$
- DC gain: 1 (no scaling)

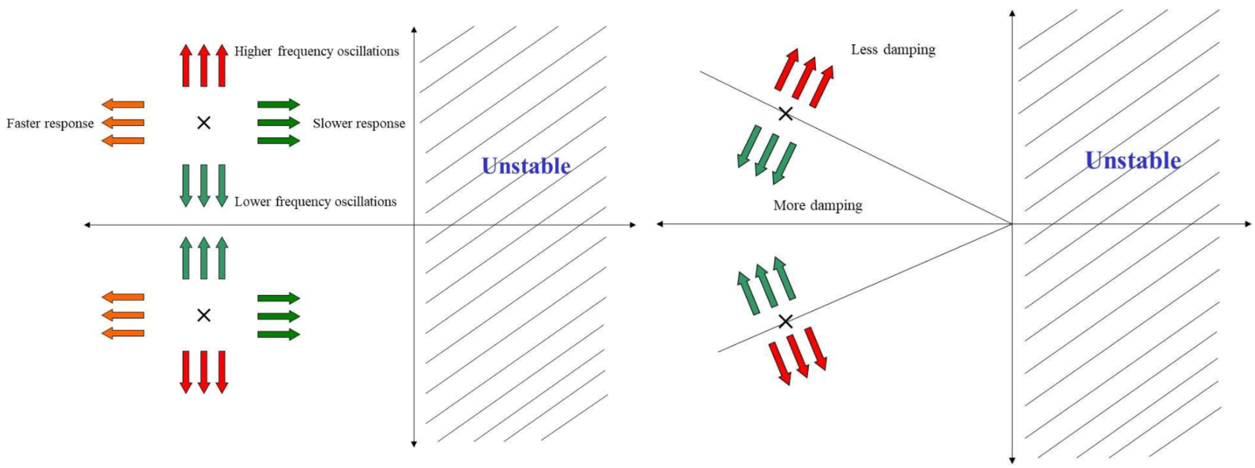

Figure 4: Response based on pole location.

- Peak time: $t_p = \dfrac{n\pi}{\omega_n\sqrt{1-\zeta^2}} = \dfrac{n\pi}{\omega_d}$
- Overshoot: $M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \iff \zeta = \sqrt{\dfrac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}}$
  - $\zeta = 0.5$ gives 16% overshoot, $\zeta = 0.7$ gives 5% overshoot
- Rise time: $t_r \approx \dfrac{1.8}{\omega_n}$ (for $\zeta = 0.5$)
- Settling time: $t_s \approx \dfrac{4.6}{\zeta\omega_n} = \dfrac{4.6}{\sigma}$

General Trends

- LHP zero: faster response; shorter rise time, larger overshoot; no effect on steady-state
- RHP zero (nonminimum-phase): longer rise time, less overshoot than LHP zero (but still increased); can cause system to start in the wrong direction
- Additional poles: slower system; longer rise time, less overshoot
- Poles and zeros with a real part more than 4 times further can be ignored

Block Diagram Simplification

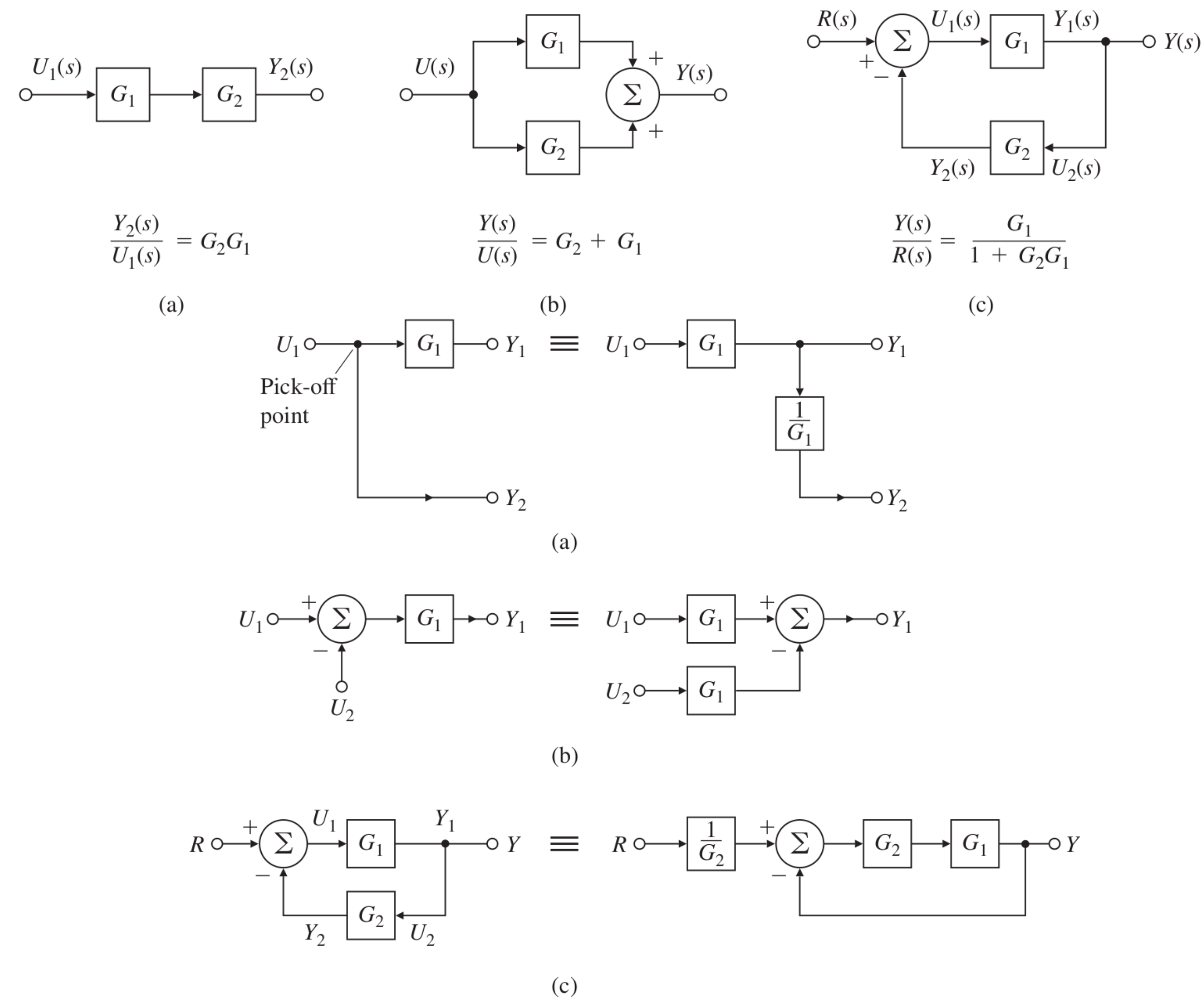

Figure 5: Block diagram reduction rules.

Control System Performance

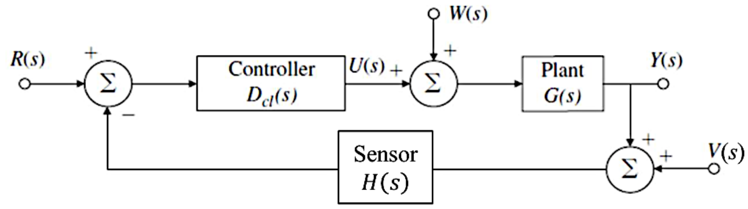

Figure 6: Closed-loop controller with disturbances.

- $Y_{cl} = \mathcal{T}R(s) + G\mathcal{S}W(s) - H\mathcal{T}V(s) \implies E_{cl} = R - Y_{cl} = (1-\mathcal{T})R - G\mathcal{S}W + H\mathcal{T}V$
  - $\mathcal{T}(s) = \dfrac{Y(s)}{R(s)} = \dfrac{G(s)D_{cl}(s)}{1+H(s)G(s)D_{cl}(s)} = T_{cl}(s)$ (assume $W(s) = V(s) = 0$)
  - $\dfrac{Y(s)}{W(s)} = G(s)\cdot\dfrac{1}{1+H(s)G(s)D_{cl}(s)} = G(s)\mathcal{S}(s)$ (assume $R(s) = V(s) = 0$)
  - $\dfrac{Y(s)}{V(s)} = -H(s)\cdot\dfrac{D_{cl}(s)G(s)}{1+H(s)D_{cl}(s)G(s)} = -H(s)\mathcal{T}(s)$ (assume $R(s) = W(s) = 0$)
  - For unity feedback: $\mathcal{T}(s) + \mathcal{S}(s) = 1 \implies E_{cl} = \mathcal{S}R(s) - G\mathcal{S}W(s) + \mathcal{T}V(s)$
- Sensitivity: $\mathcal{S}^{\mathcal{T}}G = \dfrac{G}{\mathcal{T}}\dfrac{\mathrm{d}\mathcal{T}}{\mathrm{d}G} = \dfrac{1}{1+HGD_{cl}}$
- Type: the maximum order k of a polynomial reference $r(t) = t^k$ that the system can follow with e_{ss} being constant
  - For tracking and $H(s) = 1, W(s) = V(s) = 0$, the type is the number of poles $GD_{cl}(s)$ has at $s = 0$
  - For regulation and $R(s) = V(s) = 0$, the type is the number of zeros $\dfrac{E_{cl}(s)}{W(s)} = -T_w(s) = -\dfrac{G(s)}{1+H(s)G(s)D_{cl}(s)}$ has at $s = 0$
- Error constants (wrt tracking):
  - Type 0: $K_p = \lim\limits_{s\to 0} GD_{cl}(s) \implies e_{ss} = \dfrac{1}{1+K_p}$
  - Type 1: $K_v = \lim\limits_{s\to 0} sGD_{cl}(s) \implies e_{ss} = \dfrac{1}{K_v}$
  - Type 2: $K_a = \lim\limits_{s\to 0} s^2GD_{cl}(s) \implies e_{ss} = \dfrac{1}{K_a}$

Root Locus Design Method

- Evans form: $1 + D_c(s)G(s)H(s) = a(s) + Kb(s) = 0 \implies 1 + K\dfrac{b(s)}{a(s)} = 1 + KL(s) = 0$
- s_0 is on the locus if $\displaystyle\sum_{i=1}^{m} \angle(s_0 - z_i) - \sum_{i=1}^{n} \angle(s_0 - p_i) = 180° + 360°(l-1)$
- Given $L(s) = \dfrac{b(s)}{a(s)} = \dfrac{\prod_{i=1}^{n}(s - z_i)}{\prod_{i=1}^{m}(s - p_i)}$, a positive root locus follows the following rules:
  1. There are n branches each starting from the open-loop poles; m of these branches will end at the open-loop zeros of $L(s)$, while the rest go to infinity
  2. The segments of the locus on the real axis are always to the left of an odd number of real poles and zeros (on the real axis)
  3. For the $n - m$ poles that must go to infinity, their asymptotes are lines radiating from the real axis at $s = \alpha$ at angles ϕ_l, where:
     - $\alpha = \dfrac{\sum_i p_i - \sum_i z_i}{n - m}$
     - $\phi_l = \dfrac{180° + 360°(l-1)}{n - m}$
     - $l = 1, 2, \ldots, n - m$ is the branch number
     - Geometrically this means that the asymptotes evenly divide the 360° and are always symmetric about the real axis; for an odd number of branches, there is always an asymptote towards the negative real axis
  4. Each branch departs at an angle of $\phi_{l,d} = \displaystyle\sum_{i} \psi_i - \sum_{i \neq l} \phi_i - 180°$ from an open-loop pole, where ψ_i are the angles from zeros to the pole, and ϕ_i are angles from the other poles to the pole
     - If the pole is repeated q times, $\phi_{l,d} = \displaystyle\sum_{i} \psi_i - \sum_{i \neq l} \phi_i - 180° - 360°(l-1)$ for $l = 1, 2, \ldots, q$
     - Angles of arrival at a zero are $\psi_{l,a} = \displaystyle\sum \phi_i - \sum_{i \neq l} \psi + 180° + 360°(l-1)$
  5. At points where branches intersect (where the characteristic polynomial has repeated roots), if q branches intersect at the point, then their departure angles are $\dfrac{180° + 360°(l-1)}{q}$ plus an offset; together the q branches arriving and q branches departing should form an array of $2q$ evenly spaced rays
     - If the intersection is on the real axis, use Rule 2 to determine the orientation, otherwise use Rule 4
  6. The breakaway/break-in points of the locus (i.e. intersection points) are among points where $\dfrac{\mathrm{d}L(s)}{\mathrm{d}s} = 0$
     - If the multiplicity of the root of $\dfrac{\mathrm{d}L(s)}{\mathrm{d}s} = 0$ is r, then the multiplicity of the corresponding root in the closed-loop characteristic equation is $q = r + 1$ (i.e. $r + 1$ branches meet)
- After desired point s_0 is found, $K = \dfrac{1}{|L(s)|} = \dfrac{|\prod_{i=1}^{n}(s_0 - p_i)|}{|\prod_{i=1}^{m}(s_0 - z_i)|} = \dfrac{\prod_{i=1}^{n}|s_0 - p_i|}{\prod_{i=1}^{m}|s_0 - z_i|}$
  - Substitute K back into $L(s) = \dfrac{1}{K}$ to solve for the other roots at this gain
- Lead, lag and notch compensators do not have poles or zeros at the origin, thus they do not change the system type
- Lead compensator: $D_c(s) = K\dfrac{s+z}{s+p}$ where $z < p$; approximates PD control: speeds up response (lowering rise time) and decreases overshoot
  1. Determine where closed-loop poles need to be to meet specifications
  2. Create a root locus with only a proportional controller
  3. If more damping is needed, choose z to be 1/4 to 1 times the desired ω_n and pick p to be 5 to 25 times z

4. If less damping is needed, decrease p; if more damping is needed, increase p and/or decrease z
   - The ratio p/z should be as low as possible (less than 25) in order to minimize the effects of noise from a derivative controller
5. When values of z and p are found so that the root locus passes through the desired region, select the value of K and check the step response
6. Add integral control or lag compensator if steady-state error requirements are not met

- Notch compensator: $D_c(s) = K\dfrac{s^2 + 2\zeta\omega_0 s + \omega_0^2}{(s + \omega_0)^2}$; captures problematic poles with its zeros, cancelling a specific unwanted resonant frequency in the plant
  - Choose the zeros close to the undesirable pole and a bit closer to the imaginary axis
  - Imaginary part of zeros is either above or below the poles, so the locus stays in the LHP
- Lag compensator: $D_c(s) = K\dfrac{s + z}{s + p}$ where $z > p$; approximates PI control, decreasing steady-state error
  1. Determine the factor of increase in the error constant needed, which gives z/p (typically between 3 to 10)
  2. Select z to be approximately 100 to 200 times smaller than the system's dominant natural frequency
  3. Plot the resulting root locus and adjust z and p as necessary
  4. Plot the step input to verify that the time domain response is still satisfactory
     - If the lag compensator is too slow, increase z and p while keeping their ratio constant, but keep still far from dominant poles

Frequency Design Method

- Bandwidth ω_{BW}: the highest frequency where the output still tracks the input in a satisfactory manner, typically when gain hits $\sqrt{2}/2 = 0.707$
  - A higher bandwidth means a faster response – the larger ω_{BW} is, the larger ω_n is and the shorter our rise and peak times
- Resonant peak M_r: the maximum value of the amplitude ratio
  - $M_r = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}, \omega_r = \omega_n\sqrt{1-2\zeta^2}$ for a second-order system
- Gain margin (GM): the factor by which K can be increased before the system becomes unstable; $GM < 1$ indicates instability
  - This is the value of $\dfrac{1}{|KG(j\omega)|}$ where $\angle G(j\omega) = -180°$
- Phase margin (PM): the amount by which $\angle G(j\omega)$ exceeds $-180°$ (less negative) when $|KG(j\omega)| = 1$; $PM < 0$ indicates instability
  - A value of $PM = 30°$ is typically regarded as the lowest value for a safe stability margin
  - $PM = \tan^{-1}\left(\dfrac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4}-2\zeta^2}}\right)$ for a second-order system
  - $\zeta \approx \dfrac{PM°}{100}$ for $PM < 65°$
- Crossover frequency ω_c: the frequency at which the open-loop magnitude is unity
  - $PM = \angle L(j\omega_c) - (-180°)$
  - $\omega_c \le \omega_{BW} \le 2\omega_c$ with $\omega_c = \omega_{BW}$ for $PM = 90°$
- Disturbance rejection bandwidth ω_{DRB}: the max frequency at which the disturbance rejection (i.e. sensitivity $\mathcal{S}$) is below a certain amount, usually -3 decibels/0.707

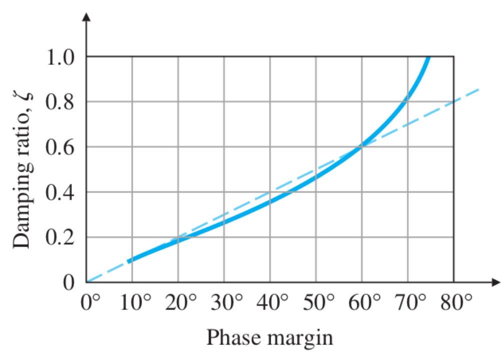

Figure 7: Relationship between ζ and PM.

- Rule of thumb: having $|KG(j\omega)|$ at a constant slope of -1 for a decade around ω_c will result in a PM of 90°
- A unity feedback system of type n has an open-loop magnitude plot with a slope of $-n$ at low frequencies
  - $|KD_cG(j\omega)| \approx \dfrac{K_n}{\omega^n}$ at low frequencies
- Lead compensator: $D_c(s) = \dfrac{T_Ds+1}{\alpha T_Ds+1}$ where $\alpha < 1$, with corner frequencies $\omega_l = \dfrac{1}{T_D}$ (low) and $\omega_h = \dfrac{1}{\alpha T_D}$ (high); increases the crossover frequency and the speed of response
  1. Determine K to satisfy error or bandwidth requirements
     - For error, pick K to satisfy the error constant
     - For bandwidth, pick K so that ω_c is within a factor of two below the desired closed-loop bandwidth
  2. Evaluate the PM of the uncompensated system using this K
  3. Find the amount of PM increase we need (add a safety margin, usually 5° or more)

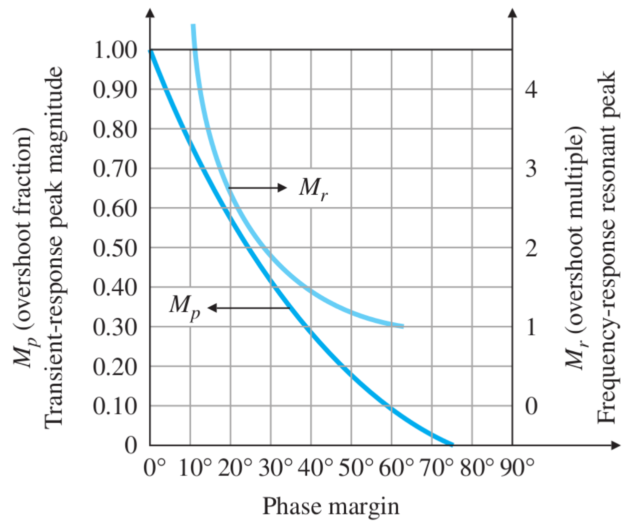

Figure 8: Relationship between M_p and M_r and PM.

4. Determine $\alpha = \dfrac{1 - \sin\phi_{max}}{1 + \sin\phi_{max}}$
5. Pick the desired crossover frequency and make ω_{max} there, and determine T_D using $\dfrac{1}{T_D} = \omega_{max}\sqrt{\alpha}$
6. Draw the compensated frequency response and check that the PM requirement is satisfied; iterate if not

- Lag compensator: $D_c(s) = \alpha \dfrac{T_I s + 1}{\alpha T_I s + 1}$, where $\alpha > 1$; decreases steady-state error; alternatively can be used to decrease the magnitude at frequencies above break points (with adjustment in K), to increase PM
  1. Determine the gain K required to get the desired PM without compensation, with a 5° to 10° margin to account for the PM reduction of the compensator
  2. Draw the Bode plot of the uncompensated open-loop TF and check the low-frequency gain, which gives the steady-state error
  3. Determine the value of α to meet the steady-state error requirement – α is how much more we need to multiply the low-frequency gain by in order to meet the steady-state error requirement
  4. Choose the upper corner frequency $\dfrac{1}{T_I}$ (the zero) to be one octave to multiple decades below the uncompensated ω_c
  5. Iterate on the design and verify that it meets requirements