Lecture 9, Feb 5, 2024

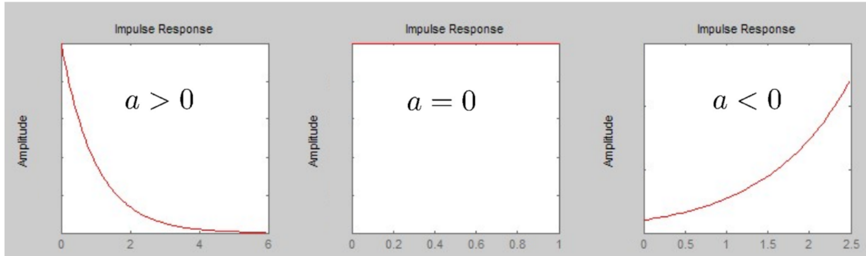
First-Order System Response

- Consider a pure integrator: $y(t) = \int_0^t u(t)\,\mathrm{d}t + y(0)$ which has transfer function $H(s) = \frac{1}{s}$ if $y(0) = 0$
  - The ODE is $\dot{y}(t) = u(t)$
  - The impulse response is $y_i(t) = \mathcal{L}^{-1}\left\{H(s)\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
  - The step response is $y_s(t) = \mathcal{L}^{-1}\left\{H(s)\frac{1}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$
  - What if the initial condition is not zero?
    * Laplace transform the ODE to get $sY(s) - y(0) = U(s) \implies Y(s) = \frac{1}{s}U(s) + \frac{1}{s}y(0)$
    * For a step response, $y_s(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s}y(0)\right\} = t + y(0)1(t) = t + y(0)$
- Consider an RC circuit with input voltage $Ku(t)$
  - Form the ODE: $T\dot{y}(t) + y(t) = Ku(t)$ where $T = RC$
  - Laplace transform: $TsY(s) + Y(s) = KU(s) \implies H(s) = \frac{Y(s)}{U(s)} = \frac{K}{Ts+1}$
  - Impulse response: $y_i(t) = \mathcal{L}^{-1}\left\{H(s)\right\} = \frac{K}{T}e^{-\frac{t}{T}}$
  - Step response: $y_s(t) = \mathcal{L}^{-1}\left\{\frac{K}{s(Ts+1)}\right\} = K\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{T}{Ts+1}\right\} = K\left(1 - e^{-\frac{t}{T}}\right)$
  - We can see that T is the time constant of the system; the smaller it is, the faster the system evolves
  - DC gain: $y_ss = \lim_{t\to\infty} y_s(t) = \lim_{s\to 0} s\frac{K}{s(Ts+1)} = K$
- In general a first-order system has transfer function $H(s) = \frac{b}{s+a}$ and impulse response $h(t) = be^{-at}1(t)$
  - For positive a, this is stable and the system decays to 0; for negative a, this is unstable; for $a = 0$ the system maintains a constant output
    * Positive a gives poles in the LHP and negative a gives poles in the RHP
  - The step response is given by $y_s = \frac{b}{a}(1 - e^{-at})1(t)$
    * For positive a, this converges to the DC gain $\frac{b}{a}$
    * For negative a this diverges exponentially
    * For zero a this gives a linear response (note we can derive this by nothing $H(s) = \frac{b}{s}$ in this case)
- The time constant is given by $T = \frac{1}{a}$
  - The rise time is given by $t_r \approx 2.2T$, which is the time taken for the output to go from 10% to 90% of the DC gain
  - The settling time is given by $t_s \approx \frac{4.6}{a}$, which is the time taken for the output to reach 99% of the DC gain
- In a first-order system, there is never any overshoot or oscillation; the output never passes the steady state value

Second Order System Response

- Consider a spring-mass-dashpot system: $m\ddot{y}(t) + b\dot{y}(t) + ky(t) = kf(t)$
  - Laplace transform: $m(s^2Y(s) - sy(0^-) - \dot{y}(0^-)) + b(sY(s) - y(0^-)) + kY(s) = kF(s)$
  - $Y(s) = \frac{\frac{k}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}}F(s) + \frac{s + \frac{b}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}}y(0^-) + \frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}}\dot{y}(0^-)$

➢ Impulse Response: $h(t) = \mathcal{L}^{-1}[H(s)] = b\,e^{-at}\,1(t)$



➢ Step Response: $y_{step}(t) = \mathcal{L}^{-1}[H(s)/s] = \dfrac{b}{a}(1 - e^{-at})1(t)$

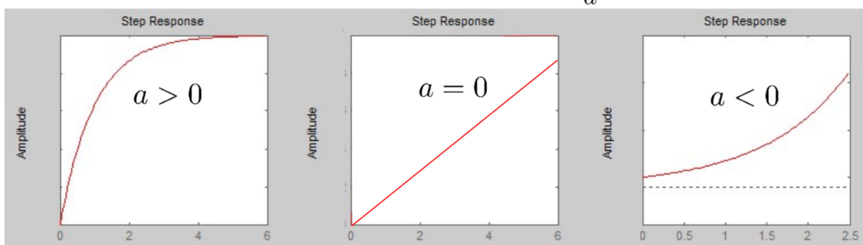


Figure 1: Behaviour of the impulse and step responses for a general (strictly proper) first-order system.

- Assuming zero state, $H(s) = \dfrac{Y(s)}{F(s)} = \dfrac{\frac{k}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}} = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
  * $\omega_n = \sqrt{\dfrac{k}{m}}$ is the natural frequency
  * $\zeta = \dfrac{b}{2\sqrt{km}}$ is the damping ratio
- The poles are at $-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$
  * Depending on ζ we can get real or imaginary poles
- If $\zeta > 1$ (i.e. $b > 2\sqrt{km}$) we have two distinct real poles; the system is *overdamped*
  * Let $-\sigma_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$, $-\sigma_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$
  * Then $\omega_n = \sqrt{\sigma_1\sigma_2}, \zeta = \dfrac{\sigma_1 + \sigma_2}{2\sqrt{\sigma_1\sigma_2}}$
- If $\zeta = 1$ (i.e. $b = 2\sqrt{km}$) we have two overlapping real poles; the system is *critically damped*
  * $H(s) = \dfrac{\sigma^2}{(s + \sigma)^2}$ where $\sigma = \omega_n$
- If $0 \leq \zeta < 1$ (i.e. $b < 2\sqrt{km}$) we have two complex conjugate poles; the system is *underdamped*
  * The poles are $s_1, s_2 = -\sigma \pm j\omega_d$ where $\sigma = \zeta\omega_n, \omega_d = \omega_n\sqrt{1 - \zeta^2}$
  * $H(s) = \dfrac{\omega_n^2}{(s - (-\sigma + j\omega_d))(s - (-\sigma - j\omega_d))} = \dfrac{\sigma^2 + \omega_d^2}{(s + \sigma)^2 + \omega_d^2}$
  * In this case the system oscillates
  * ω_d is the oscillation frequency and σ is the decay rate

• Consider the impulse response of the underdamped case
  - $y_i(t) = \mathcal{L}^{-1}\left\{\dfrac{(\sigma^2 + \omega_d^2)}{(s + \sigma)^2 + \omega_d^2}\right\} = \mathcal{L}^{-1}\left\{\dfrac{(\sigma^2 + \omega_d^2)}{\omega_d}\dfrac{\omega_d}{(s + \sigma)^2 + \omega_d^2}\right\} = \dfrac{\sigma^2 + \omega_d^2}{\omega_d}e^{-\sigma t}\sin(\omega_d t)$
  - Alternatively $y_i(t) = \dfrac{\omega_n}{\sqrt{1 - \zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1 - \zeta^2}t\right)$
  - The response is a decaying exponential

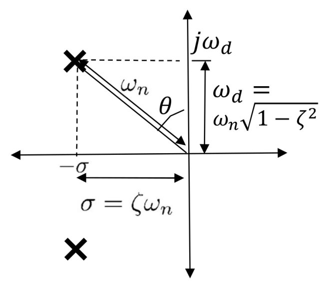


Figure 2: Illustration of the system variables in polar form.

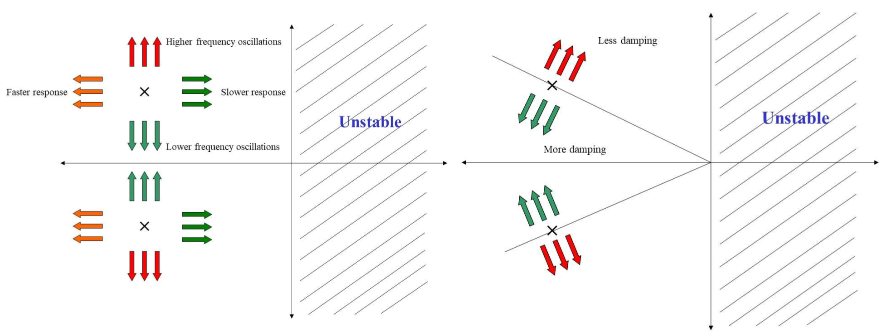


Figure 3: Response of an underdamped second-order system based on pole location.

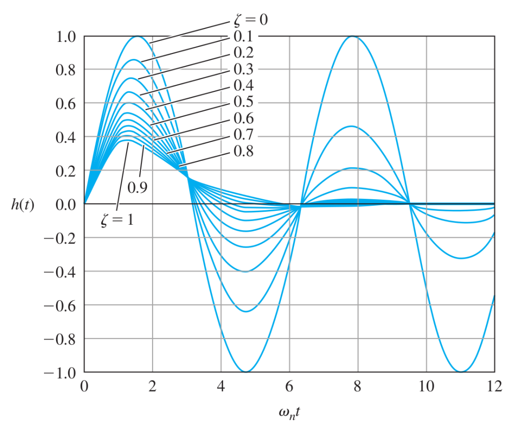


Figure 4: Impulse response of an underdamped second-order system.