Lecture 7, Jan 29, 2024

Laplace Transform

Definition

The *Laplace transform* for a generic function $f(t)$ is defined as

$$F(s) \equiv \int_{-\infty}^{\infty} f(t)e^{-s}\,\mathrm{d}t$$

The *unilateral* (one-sided) Laplace transform is defined as

$$F(s) = \mathcal{L}\left\{f(t)\right\} \equiv \int_{0^-}^{\infty} f(t)e^{-st}\,\mathrm{d}t$$

where $s = \sigma + j\omega$ is a complex frequency variable with units of inverse time.

- The Laplace transform transforms linear ODEs into algebraic equations
- For our purposes since we only consider $t \geq 0$, we consider all functions to be 0 for $t < 0$ and so the unilateral transform suffices
- $F(s)$ exists (i.e. the integral converges) if for all $\mathrm{Re}(s) > \alpha$ we have $|f(t)| < Me^{\alpha t}$ for all $s \in \mathbb{C}, M \in \mathbb{R}$, i.e. $f(t)$ grows slower than exponential
  - When multiplying transforms, the output is only valid for values of s in the intersection of the regions of convergence
- Some examples:
  - Unit step: $\mathcal{L}\left\{1(t)\right\} = \int_0^\infty 1(t)e^{-st}\,\mathrm{d}t = -\frac{1}{s}\left[e^{-st}\right]_0^\infty = \frac{1}{s}$
  - Unit impulse: $\mathcal{L}\left\{\delta(t)\right\} = \int_{0^-}^\infty \delta(t)e^{-st}\,\mathrm{d}t = e^{-st}\big|_{t=0} = 1$
    - Note that we had to start at 0^- to include 0 in the integration region
  - Exponential: $\mathcal{L}\left\{e^{\alpha t}\right\} = \int_0^\infty e^{\alpha t}e^{-st}\,\mathrm{d}t = \int_0^\infty e^{-(s-a)t}\,\mathrm{d}t = -\frac{1}{s-a}\left[e^{-(s-a)t}\right]_0^\infty = \frac{1}{s-a}$
    - Note we need to assume $\mathrm{Re}(s) > \mathrm{Re}(a)$ so that the exponent has a negative real part
  - Sinusoid:
$$\begin{aligned}\mathcal{L}\left\{\cos(\omega t)\right\} &= \int_0^\infty \cos(\omega t)e^{-st}\,\mathrm{d}t \\ &= \int_0^\infty \frac{e^{j\omega t} + e^{-j\omega t}}{2}e^{-st}\,\mathrm{d}t \\ &= -\frac{1}{2(s-j\omega)}\left[e^{-(s-j\omega)t}\right]_0^\infty - \frac{1}{2(s+j\omega)}\left[e^{-(s+j\omega)t}\right]_0^\infty \\ &= \frac{1}{2(s-j\omega)} + \frac{1}{2(s+j\omega)} \\ &= \frac{s}{s^2+\omega^2}\end{aligned}$$
    - Similarly we can show $\mathcal{L}\left\{\sin(\omega t)\right\} = \frac{\omega}{s^2+\omega^2}$
  - Power of t:
$$\begin{aligned}\int_0^\infty t^n e^{-st}\,\mathrm{d}t &= \left[-\frac{t^n}{s}e^{-st}\right]_0^\infty + \int_0^\infty nt^{n-1}\frac{e^{-st}}{s}\,\mathrm{d}t \\ &= \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,\mathrm{d}t \\ &= \frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\} \\ &= \frac{n!}{s^{n+1}}\end{aligned}$$

    * Therefore the unit ramp function has $\mathcal{L}\{t\} = \frac{1}{s^2}$

- Important properties:
  - Linearity/superposition: $\mathcal{L}\{\alpha_1 f_1(t) + \alpha_2 f_2(t)\} = \alpha_1 F_1(s) + \alpha_2 F_2(s)$
  - Time delay:
$$\begin{aligned}\mathcal{L}\{f(t-\tau)1(t-\tau)\} &= \int_0^\infty f(t-\tau)1(t-\tau)e^{-st}\,\mathrm{d}t \\ &= \int_\tau^\infty f(t-\tau)e^{-st}\,\mathrm{d}t \\ &= \int_0^\infty f(\lambda)e^{-s(\tau+\lambda)}\,\mathrm{d}\lambda \\ &= e^{-\tau s}\int_0^\infty f(\lambda)e^{-s\lambda}\,\mathrm{d}\lambda \\ &= e^{-\tau s}F(s)\end{aligned}$$
    * A delay in time domain is a multiplication by an exponential in Laplace domain
  - Differentiation:
$$\begin{aligned}\mathcal{L}\left\{\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right\} &= \int_0^\infty e^{-st}\dot{f}(t)\,\mathrm{d}t \\ &= \left[f(t)e^{-st}\right]_0^\infty + s\int_0^\infty f(t)e^{-st}\,\mathrm{d}t \\ &= sF(s) - f(0)\end{aligned}$$
    * Note the $f(0)$ term vanishes for a zero-state response
    * For higher derivatives: $\mathcal{L}\left\{\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t)\right\} = s(sF(s) - f(0)) - \dot{f}(0) = s^2F(s) - sf(0) - \dot{f}(0)$
    * Going backwards: $\mathcal{L}\{tf(t)\} = -\frac{\mathrm{d}}{\mathrm{d}s}F(s)$
  - Integration:
$$\begin{aligned}\mathcal{L}\left\{\int_0^t f(\tau)\,\mathrm{d}\tau\right\} &= \int_0^\infty \int_0^t f(\tau)\,\mathrm{d}\tau e^{-st}\,\mathrm{d}t \\ &= -\frac{1}{s}\left[\int_0^t f(\tau)\,\mathrm{d}\tau e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty f(t)e^{-st}\,\mathrm{d}t \\ &= \frac{1}{s}F(s)\end{aligned}$$
  - Convolution:
$$\begin{aligned}\mathcal{L}\{f(t) * h(t)\} &= \int_0^\infty \int_0^t f(t-\tau)h(\tau)\,\mathrm{d}\tau e^{-st}\,\mathrm{d}t \\ &= \int_0^\infty \int_0^t f(t-\tau)h(t)e^{-st}\,\mathrm{d}\tau\,\mathrm{d}t \\ &= \int_0^\infty \int_\tau^\infty f(t-\tau)h(\tau)e^{-st}\,\mathrm{d}t\,\mathrm{d}\tau \\ &= \int_0^\infty \int_0^\infty f(\lambda)h(\tau)e^{-s(\lambda+\tau)}\,\mathrm{d}\lambda\,\mathrm{d}\tau \\ &= \int_0^\infty f(\lambda)e^{-s\lambda}\,\mathrm{d}\lambda \int_0^\infty h(\tau)e^{-s\tau}\,\mathrm{d}\tau \\ &= F(s)H(s)\end{aligned}$$
    * This means we can multiply the Laplace transform of the input by the Laplace transform of the impulse response to get the Laplace transform of the output
    * Note $\mathcal{L}\{f(t)h(t)\} = \frac{1}{2\pi j}(F(s) * H(s))$
  - Final Value Theorem: $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$
    * Recall that $\mathcal{L}\left\{\frac{\mathrm{d}}{\mathrm{d}t}f\right\} = sF(s) - f(0)$
    * $\lim_{s\to\infty}(sF(s) - f(0)) = \lim_{s\to 0} sF(s) - f(0) = \lim_{s\to 0}\int_0^\infty e^{-st}\frac{\mathrm{d}f}{\mathrm{d}t}\,\mathrm{d}t = \int_0^\infty \frac{\mathrm{d}f}{\mathrm{d}t}\,\mathrm{d}t = \lim_{t\to\infty} f(t) - f(0)$

    * Note this requires that $f(t)$ and $\frac{\mathrm{d}f}{\mathrm{d}t}$ have Laplace transforms, and $\lim_{t\to\infty} f(t)$ exists, i.e. it is *stable*
  - Initial Value Theorem: $\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$

Transfer Functions

Definition

Transfer Function: The ratio of the Laplace transforms of the output to the input of a system, assuming that the system was initially at equilibrium (zero state/initial conditions).

- All transfer functions assume zero-state; if we want to look at initial conditions we shouldn't use transfer functions
- Given any input $u(t)$ to the system, the output of the system in time domain is $y(t) = h(t) * u(t)$ where $h(t)$ is the impulse response
- In Laplace domain, the output is $Y(s) = H(s)U(s)$ where $H(s) = \frac{Y(s)}{U(s)}$, the Laplace transform of the impulse response, is the transfer function
- For all LTI systems, the transfer function of the system fully characterizes the system dynamics
- Most transfer functions are rational functions $H(s) = K_H \frac{n_H(s)}{d_H(s)} = K_h \frac{s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$
  - *Poles* are the roots of $d_H(s)$
    * These are more important than the zeros
  - *Zeros* are the roots of $n_H(s)$
  - Poles are denoted with an X while zeros are denoted by O on the complex plane when plotting
  - K_H is the transfer function *gain*
  - $d_H(s)$ is the *characteristic equation* of the transfer function/system
    * The system's *order* is the degree of $d_H(s)$
- For all causal systems, the *relative degree* $n - m$ of the transfer function is always greater than or equal to zero
  - Consider $H(s) = s$; then for an input $U(s)$, we get output $Y(s) = sU(s)$, which means $y(t) = \frac{\mathrm{d}}{\mathrm{d}t}u(t)$
    * Such a system cannot be causal, because in order to determine the derivative of the input, the system needs to somehow anticipate the input's behaviour in the future
    * e.g. if we put in a sinusoid, it will be shifted to the left, which is non-causal
    * Generally, zeros tend to push the system towards non-causality by moving the response earlier in time, while poles push the system towards causality by delaying the response
  - The transfer function is a *proper ratio* (if $m < n$, then it is *strictly proper*)
  - Most systems we will study have strictly proper transfer functions
- $H(s) = K_H \frac{\prod_{i=1}^m (s - z_i)}{\prod_{i=1}^n (s - p_i)} = \left(\frac{K_H \prod_{i=1}^m z_i}{\prod_{i=1}^n p_i}\right) \frac{\prod_{i=1}^m \left(\frac{s}{z_i} - 1\right)}{\prod_{i=1}^n \left(\frac{s}{p_i} - 1\right)}$ where z_i are the zeros, p_i the are poles
- For any LTI system, the poles of a system determines its behaviour
  - Note complex poles always come in conjugate pairs
  - Any poles on the right hand plane are unstable, i.e. the output will keep growing
    * Larger real values lead to faster growth
  - Any poles on the left hand plane are convergent, i.e. output eventually settles to 0
    * More negative real values lead to faster decay
  - Poles with zero real part neither grow nor shrink in magnitude
  - Any imaginary component in the pole causes the output to oscillate
    * Larger imaginary component lead to higher oscillation frequency
- When there are multiple poles and zeros, they will interact with each other and lead to more interesting behaviour

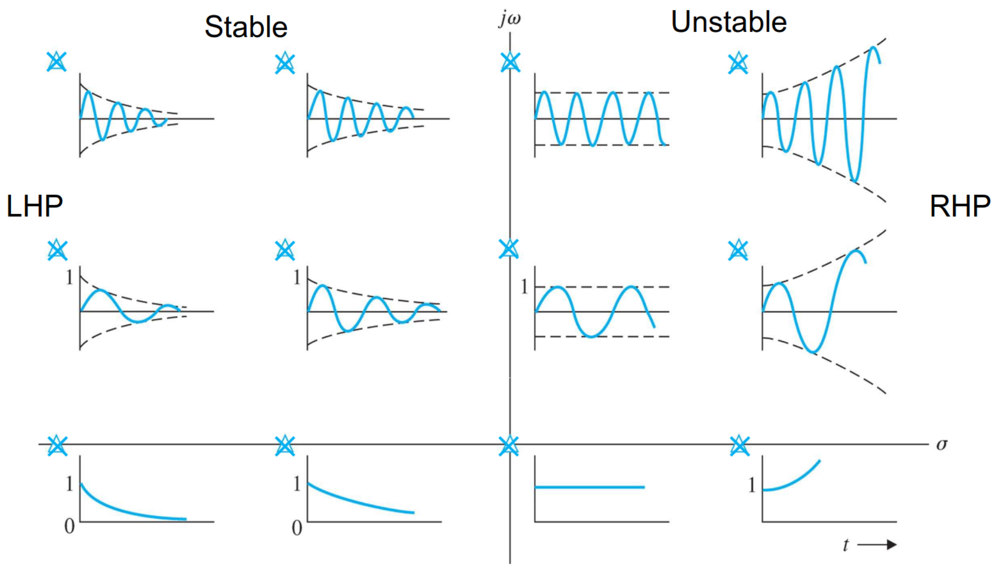

Figure 1: Behaviour of a system according to its poles.

- The *DC gain* (or *static gain*) is the steady-state response of the system to the unit step input
  - This will give an output $Y_s(s) = H(s)U(s) = \frac{1}{s}H(s)$
  - Using FVT, $\lim_{t\to\infty} y_s(t) = \lim_{s\to 0} s\left(\frac{1}{s}H(s)\right) = \lim_{s\to 0} H(s)$
  - This makes the DC gain very easy to find