Lecture 6, Jan 25, 2024

Linear Time-Invariant Systems

- *Zero state response*: the response of a system to some input when the system is initially "at rest", i.e. all inputs, outputs, states and their derivatives are initially zero
  - When we talk about linear systems, we are usually assuming zero-state
- The most important property of linear systems is homogeneity and superposition – we can scale and add inputs and the outputs will scale and add accordingly
- In a time-invariant system the parameters C are constant in time, so delaying the input will delay the output by the same amount and leave it otherwise unchanged
  - This also works in reverse – if the system output remains the same but delayed when the input is delayed, then the system is time-invariant (we can show that this implies that C is constant)
- These properties let us determine the response of a system to any general input by only knowing its impulse response
  - Any general input $u(t)$ can be approximated by a series of pulses $p_\Delta(t) = \begin{cases} \frac{1}{\Delta} & 0 \leq t \leq \Delta \\ 0 & \text{otherwise} \end{cases}$
    - The input at $t = k\Delta$ has a value $u(t) = u(k\Delta)$, so we can approximate this as $u(k\Delta) \cdot \Delta \cdot p_\Delta(t - k\Delta)$
      - Note we multiply by Δ so the integral remains the same
    - If the system has a response $h_\Delta(t)$ to $p_\Delta(t)$, then due to homogeneity and time-invariance the response to the above input is $y(t) = y(n\Delta) = u(k\Delta) \cdot \Delta \cdot h_\Delta(n\Delta - k\Delta)$
    - Then the total response to all the pulses is $y(t) = \sum_{k=0}^{\infty} u(k\Delta) \cdot \delta \cdot h_\Delta(t - k\Delta)$
  - In the limit, $p_\Delta(t)$ becomes the Dirac delta function $\delta(t)$ (or *unit impulse function*); $h_\Delta(t)$ becomes the *impulse response* $h(t)$
  - Therefore the output is a convolution: $y(t) = \int_0^\infty u(\tau)h(t - \tau)\,\mathrm{d}\tau = u(t) * h(t)$
    - Formally the convolution integral should be from $-\infty$, however we consider the zero-state response so we don't need to consider $t < 0$
    - Furthermore, if $t - \tau < 0$, we would be considering negative time for $h(t)$, which makes no sense for a causal system (in other words $y(t)$ would depend on values of the input in the future); therefore our upper bound is t instead of ∞
- Note this only applies to LTI systems, or upon linearization assuming a small input region

Summary

The response of an LTI system to any arbitrary input $u(t)$ is given by

$$y(t) = \int_0^t u(\tau)h(t - \tau)\,\mathrm{d}\tau = u(t) * h(t)$$

where $h(t)$ is the response of the system to the unit impulse $\delta(t)$.

- Note convolution has the following properties:
  - Commutativity: $x_1(t) * x_2(t) = x_2(t) * x_1(t)$
    - Obtained by a simple change of variables
  - Associativity: $x_1(t) * [x_2(t) * x_3(t)] = [x_2(t) * x_2(t)] * x_3(t)$
  - Distributivity: $x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_2(t) * x_3(t)$
  - Shift: $x_1(t) * x_2(t - T) = x_1(t - T) * x_2(t)$
    - $x_1(t) * x_2(t) = y(t) \implies x_1(t - T_1) * x_2(t - T_2) = y(t - T_1 - T_2)$
  - Impulse: $x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau) = x(t)$
  - Width: the convolution of a function covering a length of time T_1 and another function covering

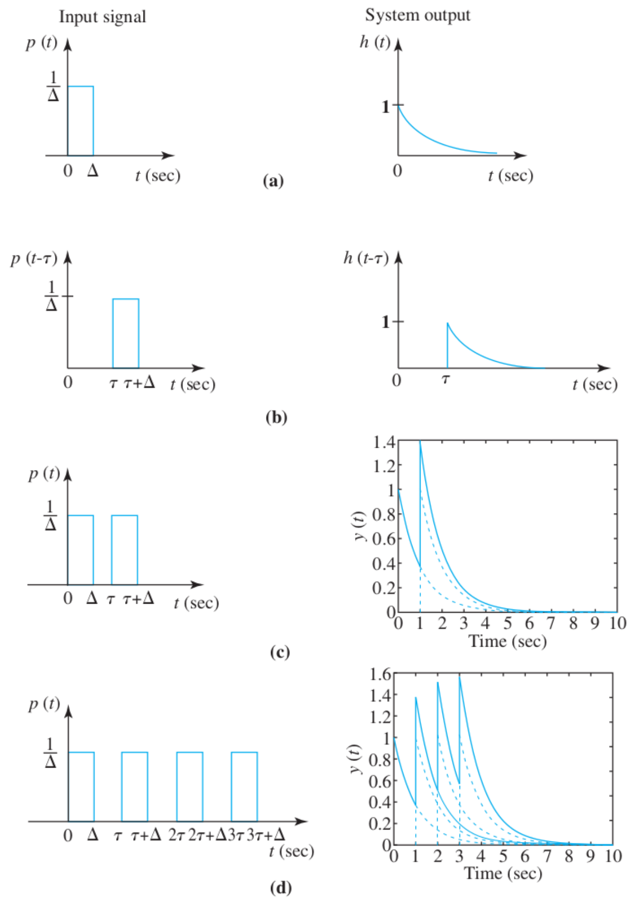

Figure 1: Response of the system to a series of pulses.

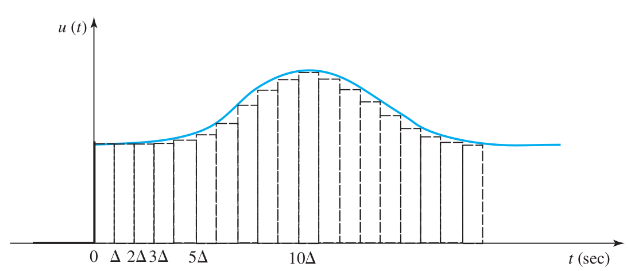

Figure 2: Approximation of any input function as a series of impulses.

T_2 covers a time of $T_1 + T_2$

- Example: find the impulse response of the following system, with $y(0^-) = 0$: $\dot{y} + ky = u(t)$
  - $\int_{0^-}^{0^+} \dot{y}\,\mathrm{d}t + k\int_{0^-}^{0^+} y\,\mathrm{d}t = \int_{0^-}^{0^+} \delta(t)\,\mathrm{d}t$
    * The second term goes to zero since y is a continuous function
    * The right hand side is by definition 1
  - $\int_{0^-}^{0^+} \dot{y}\,\mathrm{d}t = 1 \implies y(0^+) - y(0^-) = 1 \implies y(0^+) = 1$
  - Now we use the model of the system to find other times, which gives $y = Ae^{\alpha t}$
    * $A\alpha e^{\alpha t} + kAe^{\alpha t} = 0 \implies \alpha = -k$
    * $y(0^+) = 1 \implies A = 1$
  - This gives $y(t) = h(t) = e^{-kt}1(t)$ where $1(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$ is the Heaviside step function (sometimes denoted $u(t)$)
    * We need the $1(t)$ because for $t < 0$ we assumed zero-state
  - For a general input $u(t)$, $y(t) = \int_0^\infty e^{-k\tau}u(t-\tau)\,\mathrm{d}\tau$ or $\int_0^t e^{-k\tau}u(t-\tau)\,\mathrm{d}\tau$ for a causal system
    * The Heaviside function is gone because our bound starts at 0, so it is 1 for the entire integration range