Lecture 5, Jan 22, 2024

More Dynamic System Examples

- Example: pendulum with point mass m under gravity, input torque T_c around the pivot
  - $T_c - mgl\sin\theta = I\ddot{\theta} \implies \ddot{\theta} + \frac{g}{l}\sin\theta = \frac{T_c}{ml^2}$
    * We can do this since all the moments act on the same axis in planar motion
  - Linearize using $\sin\theta \to \theta$ for small angles
    * These assumptions need to be checked later
  - In free oscillation this would oscillate around 0 with $\omega_n = \sqrt{\frac{g}{l}}$
  - If T_c is constant, then this gives a constant bias angle to the oscillation

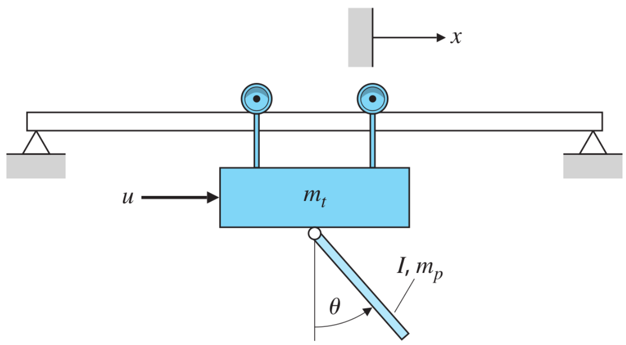

Figure 1: Example: Crane with hanging load.

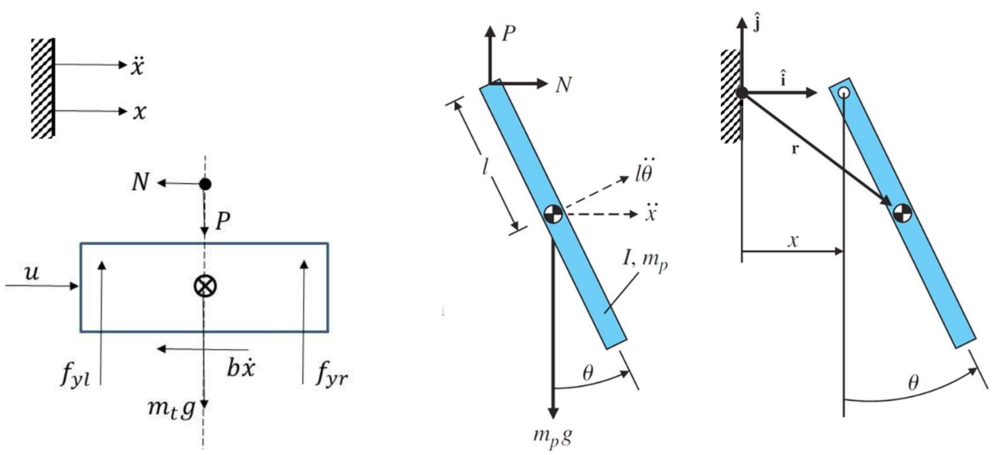

Figure 2: Free body diagram for the example.

- Example: consider a pendulum on a cart as shown above
  - The cart is subject to an applied force and viscous friction (also gravity and normal forces which cancel)
    * The pendulum also applies unknown forces to the cart at the hinge position
    * $u - N - b\dot{x} = m_t\ddot{x}$
  - For the pendulum we write the equations about the centre of mass for both forces and moments, since we don't have a fixed point on the body anymore
    * To find the accelerations of the centre of mass we differentiate its position
      - $\boldsymbol{r} = x\hat{\imath} + l(\hat{\imath}\sin\theta - \hat{\jmath}\cos\theta)$
      - $\ddot{\boldsymbol{r}} = \ddot{x}\hat{\imath} + l\ddot{\theta}(\hat{\imath}\cos\theta + \hat{\jmath}\sin\theta) - l\dot{\theta}^2(\hat{\imath}\sin\theta - \hat{\jmath}\cos\theta)$

    * $N = m_p\ddot{x} + m_p l\ddot{\theta}\cos\theta - m_p l\dot{\theta}^2\sin\theta$
    * $P - m_p g = m_p l\ddot{\sin}\theta + m_p l\dot{\theta}^2\cos\theta$
    * $-Pl\sin\theta - Nl\cos\theta = I\ddot{\theta}$
  - Clean up to get $\begin{cases}(m_t + m_p)\ddot{x} + b\dot{x} + m_p l\ddot{\theta}\cos\theta - m_p l\dot{\theta}^2\sin\theta = u \\ (I + m_p l^2)\ddot{\theta} + m_p gl\sin\theta + m_p l\ddot{x}\cos\theta = 0\end{cases}$
  - We can linearize this around $\theta = 0$ for a crane system, or around $\theta = \pi$ for a segway (inverted pendulum) system, assuming $\theta, \dot{\theta}$ are small

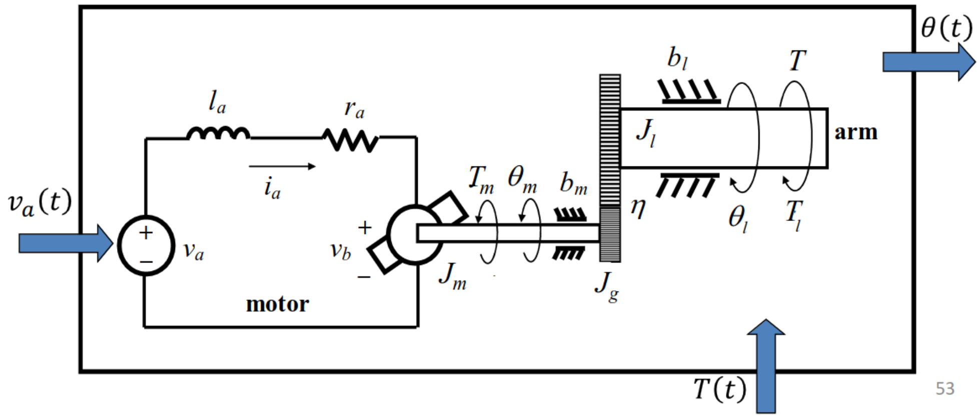

Figure 3: Mechatronics diagram of the DC brushed motor model.

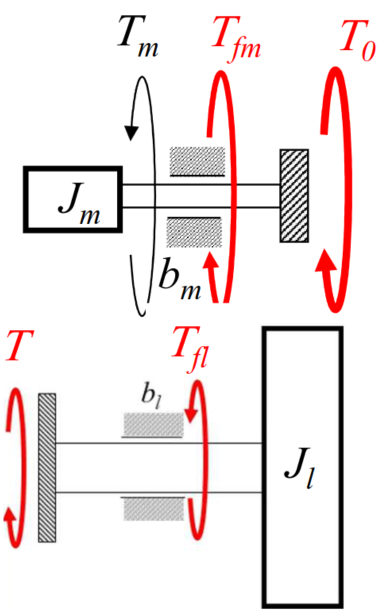

Figure 4: Free body diagrams for the two shafts.

- Example: DC brushed motor
  - The brushes power the rotor and provide the polarity switching needed to maintain a constant rotation in one direction
  - The electric part can be obtained through KVL: $l_a\frac{\mathrm{d}i_a}{\mathrm{d}t} + r_a i_a = v_a - K_e\dot{\theta}_m$
    * Lorentz law: $T_m = K_1\varphi i_a = K_l i_a$ where φ is determined by the field strength, i_a is the current and K_1 is determined by the loop geometry
    * Faraday law: $v_b = K_2\varphi\dot{\theta}_m = K_e\dot{\theta}_m$ where v_b is the back-EMF, a voltage

- The mechanical part has 2 bodies: the shaft directly connected to the rotor and the output arm, connected to the load
  * The motor generates a torque $T_m(t)$, which has to overcome a resistive torque load $T(t)$ to give the final output $\theta_m(t)$
  * For the smaller shaft: $T_m - b_m\dot{\theta}_m - T_0 = J_m\ddot{\theta}_m$
    - T_0 is some torque applied by the larger output shaft
    - $b_m\dot{\theta}_m$ is a viscous friction torque
    - J_m is the moment of inertia of the shaft
  * For the larger shaft: $T - b_l\dot{\theta}_l = J_l\ddot{\theta}_l$
  * θ_l and θ_m are related as $\theta_l = \eta\theta_m$ and $T_0 = \eta T$ since $r_m\theta_m = r_l\theta_l$
    - η is the gear ratio, $\eta = \dfrac{r_m}{r_i}$
    - Note we always use the smaller radius in the numerator so the gear ratio is always 1 or less
  * This gives $J_m\ddot{\theta}_m + b_m\dot{\theta}_m = T_m - \eta T = K_l i_a - \eta T = K_l i_a - \eta J_l\ddot{\theta}_l - \eta b_l\dot{\theta}_l$
  * Expand and rearrange: $(J_m + \eta^2 J_l)\ddot{\theta}_m + (b_m + \eta^2 b_l)\dot{\theta}_m = K_t i_a$
- Now we can solve for i_a from the mechanical system and its derivative and substitute into the first equation
- This gives us a third order linear ODE, however we can reduce this by noting that the time constant of the electric part is much smaller than the time constant of the mechanical part (i.e. after a voltage change, the current stabilizes much faster than the motor speed), which allows us to ignore the inductance of the coil and reduce the electric part to a static system, which reduces the overall differential equation to 2nd order