Lecture 4, Jan 18, 2024

Dynamic System Modelling

- Electrical, mechanical, fluid and thermal systems can be represented by analogous models, regardless of the underlying system, by taking an energy perspective
- We divide basic elements into two groups: energy storage and energy dissipation; within energy storage, elements can be either capacitive or inductive
- Each element is defined by either a *through variable* (aka *t-type*, a property that appears to flow through the element unaltered), or *across variable* (aka *a-type*, a property that is measured as a difference at the two ends of the element)
  - Capacitive elements are represented by t-type variables; inductive elements are represented by a-type variables
  - All energy dissipation elements are represented by t-type variables
- Sometimes we might want to use the integrated version of the t-type and a-type variables

System	Through Variable	Integrated Through Variable	Across Variable	Integrated Across Variable
Electrical	Current, i	Charge, q	Voltage difference, v_{21}	Flux linkage, λ_{21}
Mechanical translational	Force, F	Translational momentum, P	Velocity difference, v_{21}	Displacement difference, y_{21}
Mechanical rotational	Torque, T	Angular momentum, h	Angular velocity difference, ω_{21}	Angular displacement difference, θ_{21}
Fluid	Fluid volumetric rate of flow, Q	Volume, V	Pressure difference, P_{21}	Pressure momentum, γ_{21}
Thermal	Heat flow rate, q	Heat energy, H	Temperature difference, $\mathcal{T}_{21}$	

Figure 1: A-type and t-type state variables for the four types of systems.

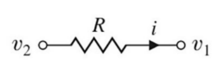
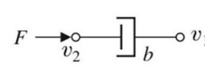
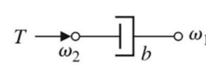
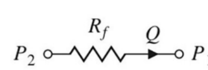
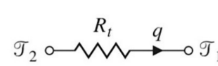
Physical Element	Governing Equation	Energy E or Power $\mathcal{P}$	Symbol
Electrical resistance	$i = \frac{1}{R} v_{21}$	$\mathcal{P} = \frac{1}{R} {v_{21}}^2$	
Translational damper	$F = b v_{21}$	$\mathcal{P} = b {v_{21}}^2$	
Rotational damper	$T = b \omega_{21}$	$\mathcal{P} = b {\omega_{21}}^2$	
Fluid resistance	$Q = \frac{1}{R_f} P_{21}$	$\mathcal{P} = \frac{1}{R_f} {P_{21}}^2$	
Thermal resistance	$q = \frac{1}{R_t} \mathcal{T}_{21}$	$\mathcal{P} = \frac{1}{R_t} \mathcal{T}_{21}$	

Figure 2: Energy dissipation elements.

- Note this is referred to as a force-current analogy; alternatively we can have a force-voltage analogy instead
- Example: cruse control model

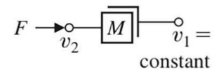
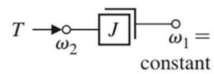
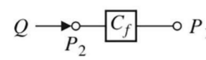
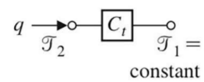
Physical Element	Governing Equation	Energy E or Power $\mathscr{P}$	Symbol
Electrical capacitance	$i = C\dfrac{dv_{21}}{dt}$	$E = \dfrac{1}{2}Cv_{21}{}^2$	
Translational mass	$F = M\dfrac{dv_2}{dt}$	$E = \dfrac{1}{2}Mv_2{}^2$	
Rotational mass	$T = J\dfrac{d\omega_2}{dt}$	$E = \dfrac{1}{2}J\omega_2{}^2$	
Fluid capacitance	$Q = C_f\dfrac{dP_{21}}{dt}$	$E = \dfrac{1}{2}C_fP_{21}{}^2$	
Thermal capacitance	$q = C_t\dfrac{d\mathcal{T}_2}{dt}$	$E = C_t\mathcal{T}_2$	

Figure 3: Capacitive energy storage elements.

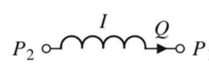
Physical Element	Governing Equation	Energy E or Power $\mathscr{P}$	Symbol
Electrical inductance	$v_{21} = L\dfrac{di}{dt}$	$E = \dfrac{1}{2}Li^2$	
Translational spring	$v_{21} = \dfrac{1}{k}\dfrac{dF}{dt}$	$E = \dfrac{1}{2}\dfrac{F^2}{k}$	
Rotational spring	$\omega_{21} = \dfrac{1}{k}\dfrac{dT}{dt}$	$E = \dfrac{1}{2}\dfrac{T^2}{k}$	
Fluid inertia	$P_{21} = I\dfrac{dQ}{dt}$	$E = \dfrac{1}{2}IQ^2$	

Figure 4: Inductive energy storage elements.

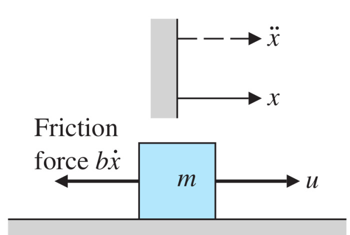


Figure 5: Free body diagram for the cruise control example.

- We apply a force u to the car of mass m, which has a resistive force proportional to the speed
- We want to know how the speed of the car varies in time
- Assumptions:
  * Car is a rigid body
  * Rotational inertia of the wheels is negligible
  * Friction/drag is proportional to speed with a factor of b
- $F = ma \implies u - b\dot{x} = m\ddot{x} \implies \ddot{x} + \frac{b}{m}\dot{x} = \frac{u}{m}$
- Change variable to v: $\dot{v} + \frac{b}{m}v = \frac{u}{m}$
- Typically, we rearrange the system to put all the outputs on the left and all the inputs on the right
- We get a first order linear ODE

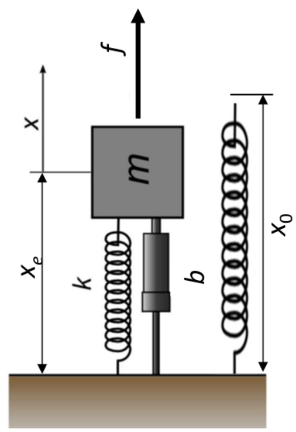


Figure 6: Mass-spring-damper example.

- Example: mass-spring-damper system
  - The input force f is applied at time 0; we want to know how x (measured from equilibrium) varies in time as a result of this force
  - x_e is the equilibrium position of the mass with no force applied; x_0 is the uncompressed length of the spring
  - In equilibrium, $k(x_0 - x_e) = mg$
  - The full FBD would have the external force m upwards, the spring force $k(x_0 - (x_e + x))$ upwards, the gravitational force mg downwards, the damping $b\dot{x}$ downwards
  - $k(x_0 - x_e - x) - b\dot{x} - mg + f = m\ddot{x}$
    * Notice that the equilibrium condition means the $k(x_0 - x_e)$ cancels with mg, so we have no g term in the final expression
  - Final ODE: $m\ddot{x} + b\dot{x} + kx = f$ (second order linear ODE)
  - In general, in mechanical systems moving around their equilibrium state, the holding (static) forces and moments required for maintaining the equilibrium do not contribute to the motion state
    * In this example, the spring force and gravity at equilibrium are the holding forces
    * Therefore we don't have x_0, x_e or g in the model
- Example: automobile suspension system
  - Each wheel of the car is equipped with a suspension system
    * The tire itself acts like a spring
    * The suspensions system consists of a spring and a dashpot
  - Consider the car moving on a road with some profile; we wish to model the vertical movement of the car body
  - This is a single input, two output system because we also need to model the movement of the wheel itself to get the movement of the car body

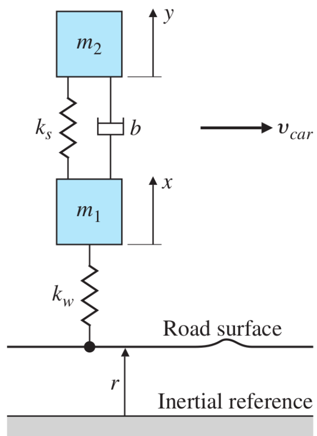


Figure 7: Automotive suspension system example.

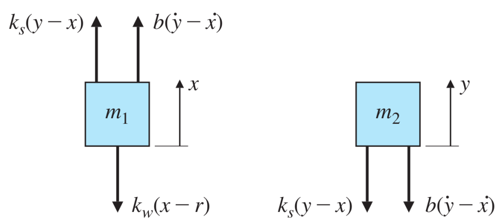


Figure 8: Free body diagram for the example.

– Drawing free body diagrams around the equilibrium allows us to ignore gravity and consider only the forces by the springs and dashpots

– Dynamic equations: $\begin{cases} b(\dot{y}-\dot{x}) + k_s(y-x) - k_w(x-r) = m_1\ddot{x} \\ -k_s(y-x) - b(\dot{y}-\dot{x}) = m_2\ddot{y} \end{cases}$

– Rearrange: $\begin{cases} \ddot{x} + \dfrac{b}{m_1}(\dot{x}-\dot{y}) + \dfrac{k_s}{m_1}(x-y) + \dfrac{k_w}{m_1}x = \dfrac{k_w}{m_1}r \\ \ddot{y} + \dfrac{b}{m_2}(\dot{y}-\dot{x}) + \dfrac{k_s}{m_2}(y-x) = 0 \end{cases}$

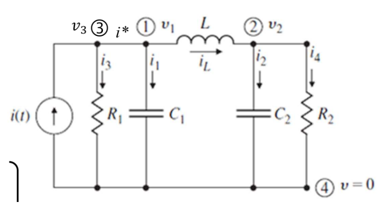


Figure 9: KCL electrical system example.

- Example: electrical system with KCL
  – Node 1: $i^* = i_1 + i_L$
  – Node 2: $i_L = i_2 + i_4$
  – Node 3: $i(t) = i_1 + i_3 + i_L$
  – $i(t) = \dfrac{v_1}{R_1} + C_1\dfrac{\mathrm{d}v_1}{\mathrm{d}t} + i_L$
  – $i_L = C_2\dfrac{\mathrm{d}v_2}{\mathrm{d}t} + \dfrac{v_2}{R_2}$

- $v_1 - v_2 = L\frac{\mathrm{d}i_L}{\mathrm{d}t}$
- $LC_1C_2\frac{\mathrm{d}^3v_2}{\mathrm{d}t^3} + \left(\frac{LC_1}{R_2} + \frac{LC_2}{R_1}\right)\frac{\mathrm{d}^2v_2}{\mathrm{d}t^2} + \left(\frac{L}{R_1R_2} + C_1 + C_2\right)\frac{\mathrm{d}v_2}{\mathrm{d}t} + \left(\frac{1}{R_1} + \frac{1}{R_2}\right) + i(t)$
- This ends up being a third order linear ODE