Lecture 25, Apr 8, 2024

Dynamic Response from Frequency Response

- For common systems, typically the open-loop transfer function has $|KG(j\omega)| \gg 1$ for $\omega \ll \omega_c$ and $|KG(j\omega)| \ll 1$ for $\omega \gg \omega_c$
  - Therefore at $\omega \ll \omega_c$, $|\mathcal{T}(j\omega)| \approx 1$, and at $\omega \gg \omega_c$, $|\mathcal{T}(j\omega)| \approx |KG(j\omega)|$
  - The magnitude of the closed-loop gain near ω_c is closely related to the phase margin
    * Note again that this peak is not exactly at ω_c
  - e.g. for $K = 1$, if $PM = 45°$, then $\angle G(j\omega_c) = -180° + PM = -135°$, and $|G(j\omega_c)| = 1$ by definition, then $|\mathcal{T}(j\omega_c)| = \left|\dfrac{G(j\omega_c)}{1 + G(j\omega_c)}\right| = \dfrac{1}{|\sqrt{(1 + \cos(-135°))^2 + \sin^2(-135°)}|} = 1.31$

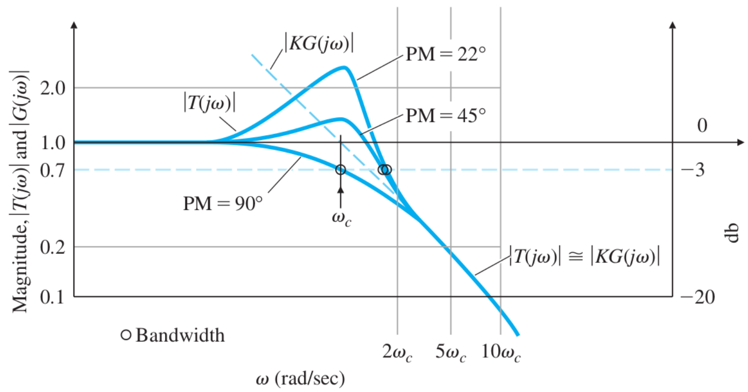

Figure 1: Closed-loop gain at ω_c for different phase margins.

- By the above calculation, $PM = 90°$, then $\omega_c = \omega_{BW}$ exactly; if $PM < 90°$, then $\omega_c \leq \omega_{BW} \leq 2\omega_c$
  - ω_{BW} is always within 1 *octave* of ω_c
  - Bandwidth is roughly equal to the natural frequency of the system, again within 1 octave
  - We typically define a closed-loop system by its bandwith and phase margin
    * $\zeta \approx \dfrac{PM°}{100}$ for $PM < 65°$ and $\omega_n \approx \omega_{BW}$
- We can find system type in the frequency response; for a unity feedback system:
  - A type 0 system's open-loop magnitude plot starts with a slope of 0 at low frequencies
    * To have a slope of 0 at low frequencies means our class 1 term has a power of $n = 0$, so it does not contribute an initial slope, so this means no poles at the origin and thus type 0
    * The low-frequency gain, K_0, is equal to the position constant K_p, since $K_p = \lim_{s \to 0} KD_cG(s) = \lim_{\omega \to 0} |KD_cG(j\omega)|$
    * We can control the steady-state error by controlling the gain K – we are shifting the entire plot up or down, which changes the low-frequency gain
  - A type 1 system's open-loop magnitude plot starts with a slope of -1
    * Now $K_v = \lim_{s \to 0} sKD_cG(s) = \lim_{\omega \to 0} \omega|KD_cG(j\omega)|$ so at low frequencies, $|KD_cG(j\omega)| \approx \dfrac{K_v}{\omega}$
    * We can find K_v by going to $\omega = 1$ and finding the intersection of the initial asymptote with the vertical line $\omega = 1$
  - A type 2 system's open-loop magnitude plot starts with a slope of -2
    * At low frequencies, $|KD_cG(j\omega)| \approx \dfrac{K_a}{\omega^2}$
    * Similarly, we can find K_a by finding the intersection of the initial slope -2 asymptote with the vertical line $\omega = 1$

Lead Compensator Design

- Consider a PD controller, $D_c(s) = (T_D s + 1)$, which is added to improve stability and dynamic response
  - This is a numerator class 2 term, which steps up the slope of the magnitude plot at the break point $\frac{1}{T_D}$
  - This essentially increases ω_c, which increases ω_{BW} and therefore ω_n which speeds up the system
  - This also increases the phase (since it's a denominator term), which increases the phase margin, which increases damping
  - Shortcomings:
    * At low frequencies the gain is 1, so this doesn't do much to the steady-state response
    * At high frequencies (i.e. noise), the gain is very high, so the noise is amplified
      - Instead, we often use a lead compensator, which has a gain that flattens at higher frequencies, to avoid noise amplification
  - Specify the break point so that the amount of increased phase desired happens near the crossover, so we can increase the PM
    * From the design requirements and the Bode plot of the uncompensated system, we can see how much additional PM we need

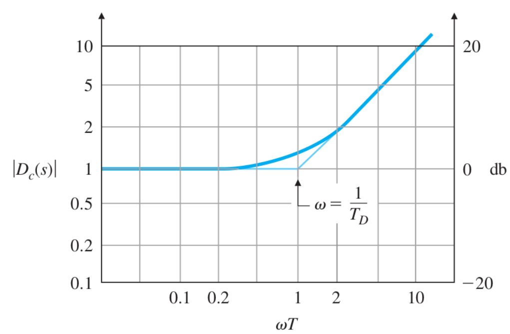

Figure 2: Bode magnitude plot for PD control.

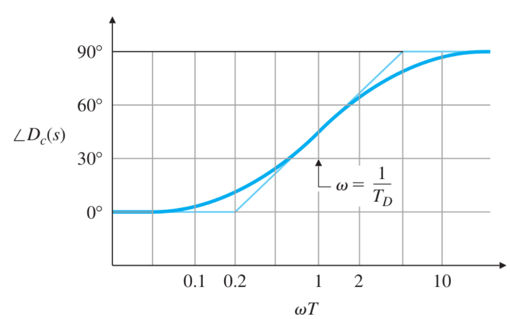

Figure 3: Bode phase plot for PD control.

- Practically, we use a lead compensator, $D_c(s) = \frac{T_D s + 1}{\alpha T_D s + 1}$ where $\alpha < 1$, with corner frequencies $\omega_l = \frac{1}{T_D}$ (low) and $\omega_h = \frac{1}{\alpha T_D}$ (high)
  - The additional denominator class 2 term steps the slope down at higher frequencies (so the magnitude plot becomes flat), so we avoid amplifying high frequency noise
  - This comes at the cost of having the phase going up and then back down (instead of staying at $+90°$ like the PD controller); the corner frequencies need to be chosen carefully so we get the maximum amount of increase to the PM
    * We typically choose $\omega_h \gg \omega_l$, typically $\omega_h > 5\omega_l$

  - The phase increase is $\phi = \angle D_c(j\omega) = \tan^{-1}(T_D\omega) - \tan^{-1}(\alpha T_D \omega)$
    * This gives $\phi_{max} = \tan^{-1}\left(\frac{1}{\sqrt{\alpha}}\right) - \tan^{-1}(\sqrt{\alpha})$ occurring at $\omega_{max} = \frac{1}{T_D\sqrt{\alpha}}$ (by differentiation)
    * $\sin\phi_{max} = \frac{1-\alpha}{1+\alpha} \implies \alpha = \frac{1-\sin\phi_{max}}{1+\sin\phi_{max}}$
      - This gives us a simpler form to find α from ϕ_{max}
      - In design, we decide how much ϕ_{max} to use, and then we obtain α
    * $\frac{1}{\alpha}$ is the *lead ratio*; the higher the lead ratio, the more we approach a PD compensator
      - Selecting this is a tradeoff between a desired PM (for good damping) and an acceptable level of high-frequency noise amplification
      - Rule of thumb is to have have a lead compensator contribute no more than 70° to the phase; if we need even more, a double lead compensator can be used

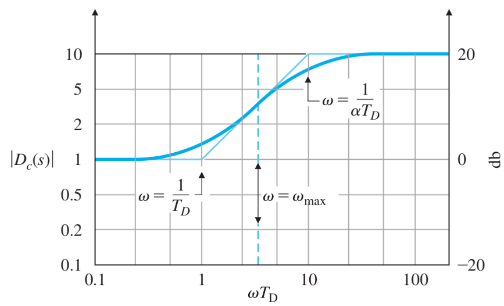

Figure 4: Bode magnitude plot for lead compensator.

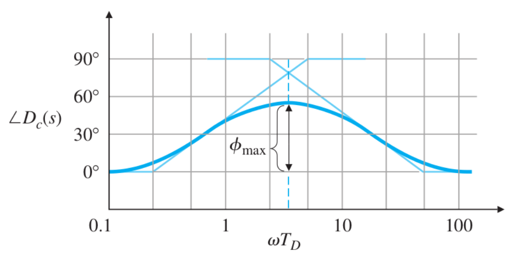

Figure 5: Bode phase plot for lead compensator.

- Both PD controller and lead compensator have no poles at the origin, so the system type is not changed
- Example: for the plant $\frac{1}{s(s+1)}$, design a lead compensator to obtain a response to a unit-ramp input with an overshoot $M_P < 25\%$ and steady-state error of no more than 0.1
  - The open-loop transfer function is type 1 (we couldn't have changed it with a lead compensator anyway)
  - Open loop TF: $L(s) = K\frac{T_D s + 1}{\alpha T_D s + 1} \cdot \frac{1}{s(s+1)}$
  - For $R(s) = \frac{1}{s^2}$, $e_{ss} = \lim_{s\to 0} \frac{1}{s + KD_c(s)\frac{1}{(s+1)}} = \frac{1}{KD_c(0)}$, therefore we need $K_v = KD_c(0) \geq 10$ from the steady-state error requirement
    * This yields a value of $K = 10$, since the lead compensator always has $D_c(0) = 1$
    * Since we don't have a lag compensator we have to use K for the steady-state response; if we had one we could save K to optimize the dynamic response
  - For $M_P < 25\%$, we use the direct relation to get $PM = 45°$
  - PM of the uncompensated system is only 20°, so we need to add more than 25°
    * The phase increase needs to be more than 25°, since the compensator zero increases ω_c due to the increase in slope, and the overall trend in phase is decreasing

    * We need to add a safety margin
  - For $\phi_{max} = 40°$ of lead, $\frac{1}{\alpha} = 5$
  - To get T_D we normally look at the desired ω_c (which influences system speed)
    * For this question we don't have a restriction on speed
    * $\frac{1}{T_D} = \omega_{max}\sqrt{\alpha}$
    * By trial and error selecting ω_{max}, we find $T_D = 0.5$
  - The final controller is $D_c(s) = 10\frac{\frac{s}{2}+1}{\frac{s}{10}+1}$

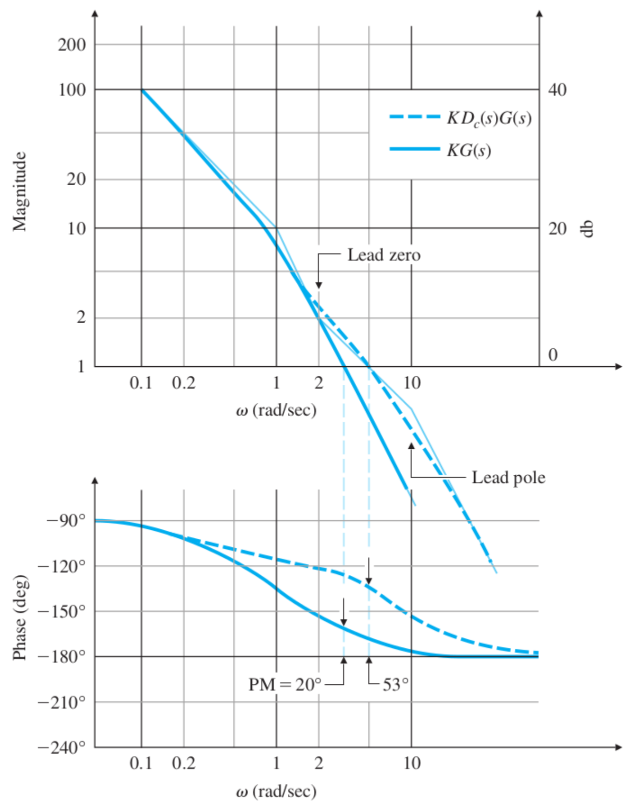

Figure 6: Bode plot of the example lead compensated system.

- For a lead compensator, we specify the parameters from design requirements as follows:
  - The crossover frequency ω_c, which determines the bandwidth hence and speed of response
  - The phase margin PM, which determines the damping ratio and overshoot
  - The low-frequency gain K, which determines the steady-state error
  - In general, lead compensation increases the ratio $\frac{\omega_c}{KD_cG(0)}$
- Design procedure for lead compensator:
  1. Determine K to satisfy error or bandwidth requirements
     - For error, pick K to satisfy the error constant
     - For bandwidth, pick K so that ω_c is within a factor of two below the desired closed-loop bandwidth
  2. Evaluate the PM of the uncompensated system using this K
  3. Find the amount of PM increase we need (add a safety margin, usually 5° or more)
  4. Determine $\alpha = \frac{1-\sin\phi_{max}}{1+\sin\phi_{max}}$
  5. Pick the desired crossover frequency and make ω_{max} there, and determine T_D using $\frac{1}{T_D} = \omega_{max}\sqrt{\alpha}$

6. Draw the compensated frequency response and check that the PM requirement is satisfied; iterate if not

- Example: type 1 servo mechanism, $KG(s) = K\dfrac{10}{s(s/2.5+1)(s/6+1)}$; design a lead compensator to obtain $PM = 45°$ and $K_v = 10$
  1. $\dfrac{1}{K_v} = \dfrac{1}{10} = \lim_{s\to 0} s \dfrac{1}{1+KD_c(s)G(s)}\dfrac{1}{s^2} \implies K = 1$
  2. Uncompensated PM is $-4°$ at $\omega_c \approx 4$
  3. We want the lead to add $\phi_{max} = 54°$ (with a safety margin of 5°)
  4. Use formulas to get $\alpha = 0.1$
  5. Choose a desired ω_c, e.g. 6 (in this case we have no hard speed requirement), giving $T_D = \dfrac{1}{\omega_c\sqrt{\alpha}} \approx 0.5$
  6. Draw the new Bode plot for $D_{c1}(s) = \dfrac{s/2+1}{s/20+1} = 10\dfrac{s+2}{s+20}$
     - We see that the PM requirement is not satisfied!
     - More iterations show that a single lead compensator cannot meet this PM requirement due to the high-frequency slope of -3
  7. Double the lead compensator; on examination this gives $PM = 46°$, meeting the requirements

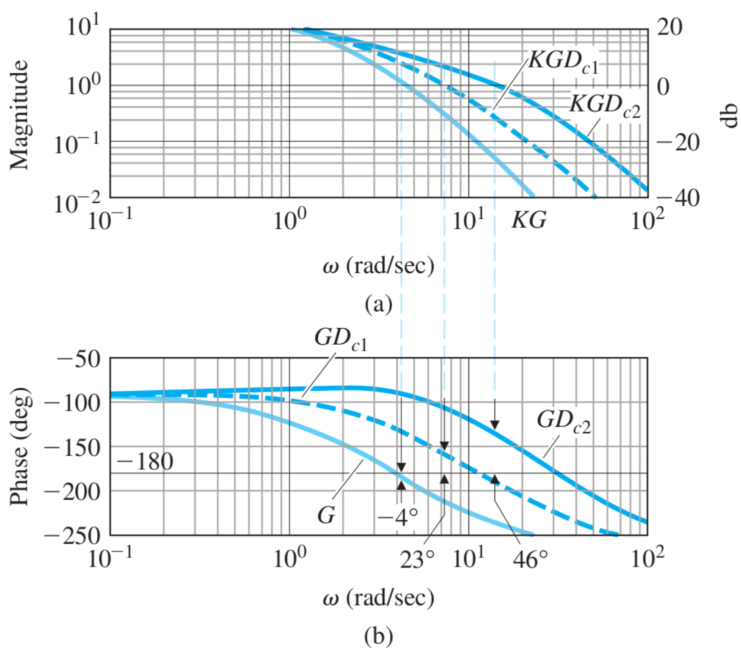

Figure 7: Bode plots for the uncompensated system, and the two iterations of lead compensators.