Lecture 24, Apr 4, 2024

System Response from Frequency Response

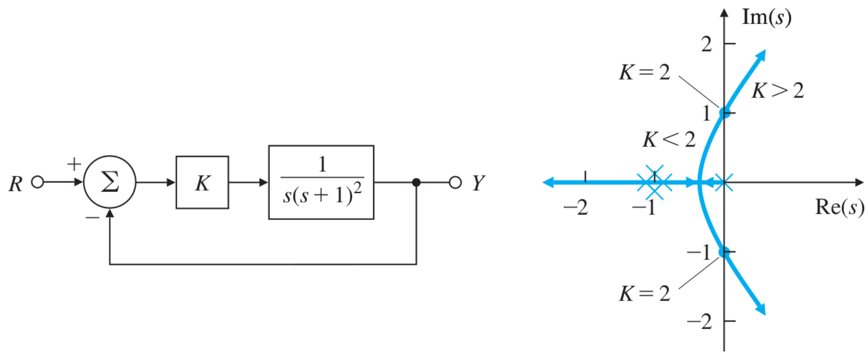

Figure 1: Typical closed-loop system and root locus.

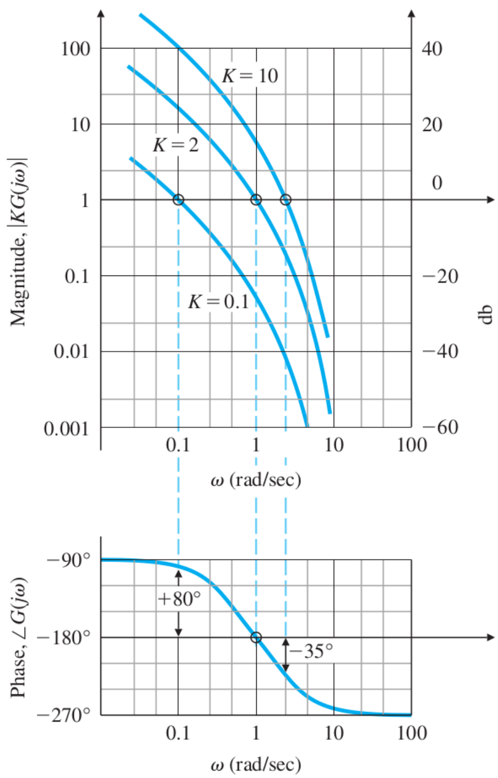

Figure 2: Open-loop Bode plot for the example system.

- Consider a unity feedback system with open-loop transfer function $L(s) = KG(s)$
- A typical root locus starts with all poles on the left hand side, and as K increases, the locus crosses the imaginary axis at some point and the system becomes unstable
- The Bode plot of $KG(j\omega_c)$ has a magnitude plot that is simply shifted vertically, and a phase plot that is identical as $G(j\omega_c)$
  - Multiplying by K increases the magnitude by a constant factor at all frequencies and has a phase of 0
- The conditions for marginal/neutral stability are $|KG(j\omega_c)| = 1$ and $\angle G(j\omega_c) = -180°$

  - These are the same conditions as having the closed-loop poles being on the imaginary axis for a root locus
  - We can look at the phase plot to see the ω_c that gives a phase of $-180°$, and then look at the value of K that gives magnitude 1 at ω_c
- For most systems, decreasing K from the neural stability value will make the system stable, while increasing it will make the system unstable
  - Therefore if $|KG(j\omega) < 1|$ at $\angle G(j\omega) = -180°$ then the system is stable; otherwise it is unstable
  - Note this does not apply if the open loop Bode plot crosses $|KG(j\omega)| = 1$ more than once
    * For such systems we need to use techniques to shift the plot so it crosses unity only once
- The degree of stability is how far we are from the value of K that gives marginal stability; we measure this through two quantities:
  - *Gain margin* (GM): the factor by which K can be increased before the system becomes unstable
    * On a Bode plot, this is how much we can move the magnitude plot up before we reach $|KG(j\omega)| = 1$
    * This is the value of $\dfrac{1}{|KG(j\omega)|}$ where $\angle G(j\omega) = -180°$
      - On a decibel scale this is the vertical distance between the value of the magnitude plot and the 0 decibel line
    * On a root locus, this is the ratio of the K value that puts the closed-loop poles on the imaginary axis and the K value that gives the poles given
    * $GM < 1$ (or negative in decibels) indicates an unstable system
  - *Phase margin* (PM): the amount by which the phase $G(j\omega)$ exceeds $-180°$ (less negative) when $|KG(j\omega)| = 1$
    * On a Bode plot, find the value of ω that gives a magnitude of 1, and the phase margin is the value of the phase at this point minus $-180°$
    * $PM < 0$ indicates an unstable system
      - A value of $PM = 30°$ is typically regarded as the lowest value for a safe stability margin
      - In design we try to go for an ideal value of $PM = 90°$ but usually we have to compromise
    * The PM for any value of K can be obtained directly from the Bode plot for $G(j\omega)$ (i.e. $K = 1$), by finding the ω that gives $|G(j\omega)| = 1/K$ and taking the phase at this frequency, subtracting $-180°$
      - This is because $|G(j\omega)| = 1/K \implies |KG(j\omega)| = 1$
      - We can also go backwards; for a value of PM, note the required ω, find the value of $|G(j\omega)|$ and take $K = 1/|G(j\omega)|$
- The (gain) *crossover frequency* ω_c is the frequency at which the open-loop magnitude is unity
  - This is highly correlated with the closed-loop bandwidth and hence the system response speed
  - $PM = \angle L(j\omega_c) - (-180°)$
- PM is more commonly used than GM in practice:
  - For a typical second order system $GM = \infty$ since phase reaches $-180°$ only at $\omega \to \infty$, at which point $|G(j\omega)| \to 0$
  - PM is also closely related to the system damping ratio
- Consider $G(s) = \dfrac{\omega_n^2}{s(s + 2\zeta\omega_n)} \implies \mathcal{T}(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$, a typical closed-loop system
  - We can derive $PM = \tan^{-1}\left(\dfrac{2\zeta}{\sqrt{\sqrt{1 + 4\zeta^4} - 2\zeta^2}}\right)$
    * $G(j\omega) = \dfrac{\omega_n^2}{(j\omega)(j\omega + 2\zeta\omega_n)}$
    * $|G(j\omega_c)| = 1 \implies \dfrac{\omega_n^2}{\omega_c\sqrt{\omega_c^2 + 4\zeta^2\omega_n^2}} = 1 \implies \omega_c^2 = -2\zeta^2\omega_n^2 \pm \sqrt{4\zeta^4\omega_n^4 + \omega_n^4} \implies \omega_c = \omega_n\sqrt{\sqrt{1 + 4\zeta^4} - 2\zeta^2}$

$$
\begin{aligned}
* \quad PM &= \angle G(j\omega_c) - (-180^\circ) \\
&= \angle \omega_n^2 - \angle (j\omega_c) - \angle (j\omega_c + 2\zeta\omega_n) + 180^\circ \\
&= 0 - 90^\circ - \tan^{-1}\left(\frac{\omega_c}{2\zeta\omega_n}\right) + 180^\circ \\
&= 90^\circ - \tan^{-1}\left(\frac{\omega_c}{2\zeta\omega_n}\right) \\
&= 90^\circ - \tan^{-1}\left(\frac{\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}}{2\zeta}\right) \\
&= \tan^{-1}\left(\frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}}\right)
\end{aligned}
$$

- For $PM < 65^\circ$, we can use a linear approximation $\zeta \approx \frac{PM^\circ}{100}$
  * This is used as a rule of thumb for other systems as well
- The resonant peak M_r and overshoot M_p can be obtained from PM as well since both are related to ζ
  * This can also serve as a rough estimate for systems other than the second-order closed-loop system we have

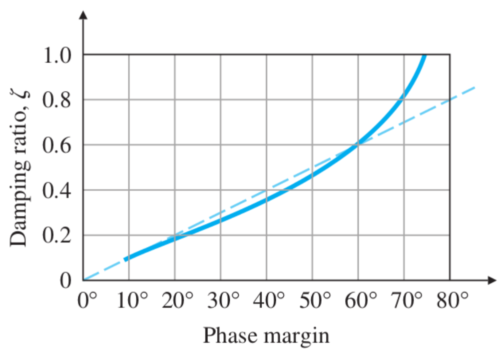

Figure 3: Relationship between ζ and PM.

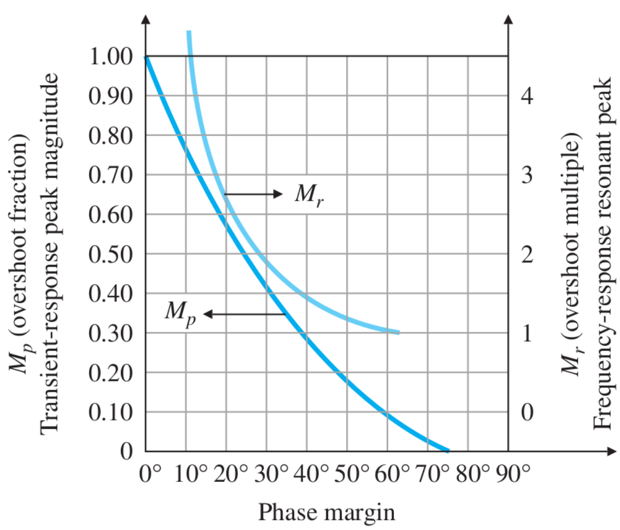

Figure 4: Relationship between M_p and M_r and PM.

- For any stable minimum phase system (i.e. no poles or zeros in the RHP), the phase of $G(j\omega)$ is uniquely related to the magnitude of $G(j\omega)$
  - $\angle G(j\omega_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}M}{\mathrm{d}u} W(u)\,\mathrm{d}u$ where $\begin{cases} M = \log\lvert G(j\omega)\rvert \\ u = \log(\omega/\omega_0) \\ W \approx \frac{\pi^2}{2}\delta(u) \end{cases}$
    * The phase is related to the slope of the magnitude plot on a log-log scale, near the frequency ω_0 we want to study
    * $\delta(u)$ is a weighting function (plot shown below)
      - This applies a much higher weight to values near $u = 0$
      - Even though the integral goes to infinity on both sides, the weighting makes it insignificant
  - If the slope of the gain is nearly constant around ω_0, we can take out $\frac{\mathrm{d}M}{\mathrm{d}u}$
  - $\angle G(j\omega_0) \approx \frac{\pi}{2}\frac{\mathrm{d}M}{\mathrm{d}u} = n \cdot 90°$ if $\frac{\mathrm{d}M}{\mathrm{d}u}$ is constant for a decade around ω_0

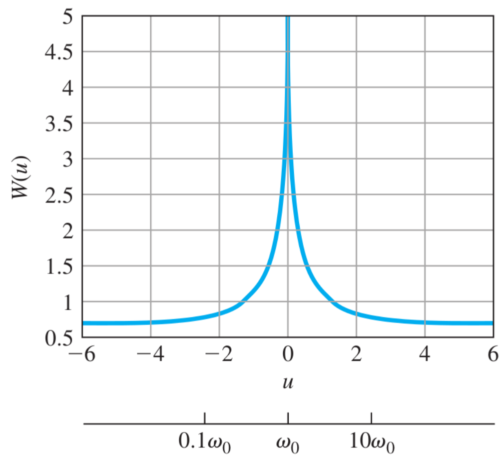

Figure 5: Plot of the weighting function.

- This means that if we can manage $\lvert KG(j\omega)\rvert$ to have a constant slope of -1 for a decade around the crossover frequency ω_c (i.e. where $\lvert KG(j\omega)\rvert = 1$), we will get a phase of $-90°$ at ω_c, which gives a PM of 90°, guaranteeing good stability of the system and a high ζ to reduce overshoot
  - This is the rule of thumb for design
  - We can adjust the value of K to shift the plot so that the slope is -1 at unity gain, or we can add compensators to change the slope for the same value of K

Example: Spacecraft Attitude Control

- Find a suitable $KD_c(s)$ to provide $M_p < 15\%$ and a bandwidth of $0.2\,\mathrm{rad/s}$ for the plant $G(s) = \frac{1}{s^2}$ and determine the frequency where the sensitivity function $\lvert\mathcal{S}\rvert = 0.7$
  - $\frac{1}{s^2}$ is class 1, so the phase plot is a constant $-180°$, and the system is always unstable; the slope of the magnitude plot is -2 which is also not good
- We want to increase the slope, so we want to add a numerator class 2 term
  - Use a PD controller: $KD_c(s) = K(T_D s + 1)$
- Start with the bandwidth of $0.2\,\mathrm{rad/s}$ which gives us a hint for ω_c; we choose $\omega_c = 0.2$
- The break point of the controller is $\frac{1}{T_D}$
  - We need to put this break point sufficiently before ω_c, so we have a sufficiently constant slope around ω_c
  - Choose the break point to be 1/4 of ω_c, so have $\omega_1 = 0.05$ and $T_D = 20$

- Plot $|D_c G(j\omega)|$ for $K = 1$, and notice the magnitude at 0.2 – in this case we have 100
- Therefore we choose $K = \dfrac{1}{|D_c G(j\omega_c|} = 0.01$
- Validate our assumption that the bandwidth is around 0.2:
  - $|\mathcal{T}(j\omega)| = \dfrac{|KD_cG|}{|1 + KD_cG|}$
  - From the plot we can see that the bandwidth is around 0.25 (when magnitude reaches around 0.7), which is close to ω_c
- For a unity feedback system, $\mathcal{S}(s) = \dfrac{E(s)}{\Theta(s)}$ (in general $1 - \mathcal{T}(s)$)
  - We want the sensitivity function to be low at the frequencies we work with, so the system is insensitive to an error in the reference
- The *disturbance rejection bandwidth*, ω_{DRB}, is the max frequency at which the disturbance rejection (i.e. sensitivity $\mathcal{S}$) is below a certain amount, usually -3 decibels
  - We always want to maximize this

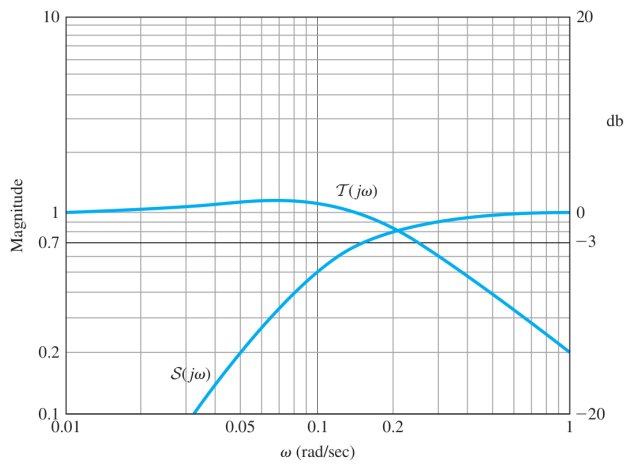

Figure 6: Bode magnitude plots of the closed-loop transfer function and sensitivity transfer function.