Lecture 23, Apr 1, 2024

Plotting Bode Plots

- Consider a general transfer function $G(s) = K\dfrac{(s+z_1)\dots(s^2+2\zeta_1\omega_{n1}s+\omega_{n1}^2)\dots}{(s+p_1)\dots(s^2+2\zeta_a\omega_{na}s+\omega_{na}^2)}$
  - z_i and p_i are real; the complex poles and zeros are in the quadratic factors, represented by their natural frequencies and damping ratios
- Rearrange as $G(s) = K_0 s^n \dfrac{(\tau_1 s+1)(\tau_2 s+1)\dots\left(\left(\frac{s}{\omega_{n1}}\right)^2+2\zeta_1\left(\frac{s}{\omega_{n1}}\right)+1\right)\dots}{(\tau_a s+1)(\tau_b s+1)\dots\left(\left(\frac{s}{\omega_{na}}\right)^2+2\zeta_a\left(\frac{s}{\omega_{na}}\right)+1\right)\dots}$
  - We factor out the poles and zeros at the origin to s^n, where n could be positive or negative
  - $\tau_1, \tau_2, \dots$ correspond to the real zeros, $\tau_a, \tau_b, \dots$ correspond to the real poles
  - $\omega_{n1}, \omega_{n2}, \dots$ and ζ_1, ζ_2 correspond to the complex zeros; $\omega_{na}, \omega_{nb}, \dots$ and ζ_a, ζ_b correspond to the complex poles
  - Substitute $s = j\omega$: $G(s) = K_0(j\omega)^n \dfrac{(j\omega\tau_1+1)(j\omega\tau_2+1)\dots\left(\left(\frac{j\omega}{\omega_{n1}}\right)^2+2\zeta_1\left(\frac{j\omega}{\omega_{n1}}\right)+1\right)\dots}{(j\omega\tau_a+1)(j\omega\tau_b+1)\dots\left(\left(\frac{j\omega}{\omega_{na}}\right)^2+2\zeta_a\left(\frac{j\omega}{\omega_{na}}\right)+1\right)\dots}$
  - This is the *Bode form* of the transfer function
- The Bode form is a composite of simpler transfer functions of the 3 classes:
  1. $K_0(j\omega)^n$ where $n \in \mathbb{Z}$
  2. $(j\omega\tau+1)^{\pm 1}$ (if the power is 1, then it is numerator class 2, while power -1 is denominator class 2)
  3. $\left(\left(\dfrac{j\omega}{\omega_n}\right)^2+2\zeta\left(\dfrac{j\omega}{\omega_n}\right)+1\right)^{\pm 1}$ (power 1 → numerator class 3; -1 → denominator class 3)
- To find the bode plot of a composite transfer function, we plot the Bode plots of each of the individual classes, and sum them up, since multiplication is addition of logs
- Class 1: $K_0(j\omega)^n$
  - Magnitude: $\log K_0|(j\omega)^n| = \log K_0 + n\log\omega$
    * The magnitude plot is a straight line with slope n (or n times 20 decibels per decade)
      - For $\omega = 1$, the value of the gain is $\log K_0$
    * For very low values of ω we will see that this is the only class that affects the slope of the Bode plot
  - Phase: $\angle K_0(j\omega)^n = n \cdot 90°$
    * The phase plot is a constant value, determined by n
- Class 2: $(j\omega\tau+1)^{\pm 1}$
  - For $\omega\tau \ll 1$, $(j\omega\tau+1)^{\pm 1} \approx 1$
  - For $\omega\tau \gg 1$, $(j\omega\tau+1)^{\pm 1} \approx (j\omega\tau)^{\pm 1}$
  - The *break point* is defined as $\omega = \dfrac{1}{\tau}$
  - Magnitude:
    * Below the break point, the gain is approximately a constant 1
    * Above the break point, the gain behaves like a class 1 term of $\tau^{\pm 1}(j\omega)^{\pm 1}$
      - The slope is a constant 1 or -1 (or ±20 decibels per decade) for this asymptote
      - The intercept is at $\tau^{\pm 1}$
    * At the break point, the gain is a factor of 1.41 (or 3 decibels) above for numerator class 2, or 0.707 (or -3 decibels) below for denominator class 2
      - At the break point, $|j\omega\tau+1|^{\pm 1} = |j+1|^{\pm 1} = \sqrt{2}^{\pm 1}$
  - Phase:
    * Below the break point, the phase is $\angle 1 = 0°$
    * Above the break point, the phase is $\angle(j\omega\tau)^{\pm 1} = \pm 90°$
    * At the break point, the phase is $\angle(j+1)^{\pm 1} = \pm 45°$

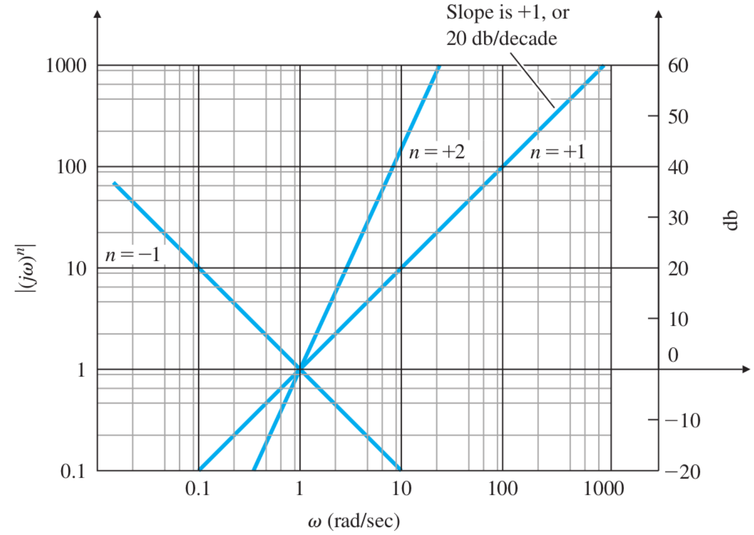

Figure 1: Magnitude plot of $(j\omega)^n$.

  * The middle asymptote intersects the lower and upper asymptotes at 5 times above and below the break point
  * At the intersection of asymptotes, the actual phase deviates from the asymptotes by about $\angle(j/5+1)^{\pm}1 = \pm 11°$
- For very low frequencies, class 2 gives a gain of 1 and phase of 0, so it has no effect on the Bode plot of the composite function
  * Rule of thumb is to ignore for ω a factor of 10 or more below the break point

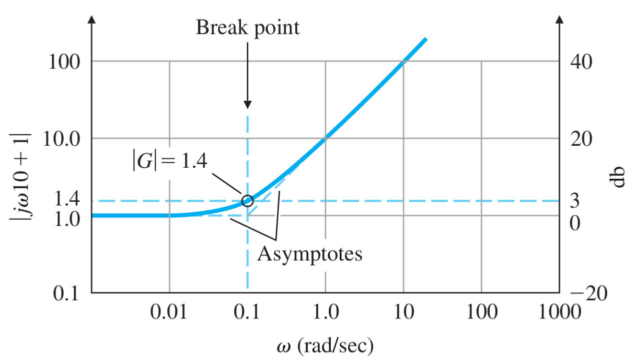

Figure 2: Magnitude plot of $(j\omega\tau + 1)$ for $\tau = 10$.

- Class 3: $\left(\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1\right)^{\pm 1}$
  - The break point is $\omega = \omega_n$
  - For $\omega \ll \omega_n$, $\left(\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1\right)^{\pm 1} \approx 1$
  - For $\omega \gg \omega_n$, $\left(\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1\right)^{\pm 1} \approx \left(\frac{j\omega}{\omega_n}\right)^{\pm 2}$
  - Magnitude:
    * For $\omega \ll \omega_n$ the gain is again approximately 1
    * For $\omega \gg \omega_n$ the gain behaves like a class 1 term of $\frac{1}{\omega_n^{\pm 2}}(j\omega)^{\pm 2}$

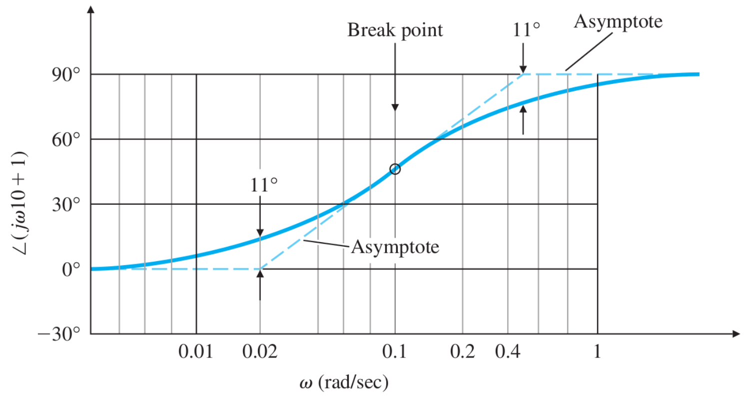

Figure 3: Phase plot of $(j\omega\tau + 1)$ for $\tau = 10$.

- The slope is a constant ± 2 (or ± 40 decibels per decade)
  * The transition between the two asymptotes depends on ζ
    - At the break point, the magnitude is a factor of $(2\zeta)^{\pm 1}$ above/below a gain of 1
      – For $\omega = \omega_n$, $(j^2 + 2\zeta + 1)^{\pm 1} = (2\zeta)^{\pm 1}$
    - For a power of $+1$ the magnitude goes down at the break point, while for -1 the magnitude goes up
  * The peak has a magnitude of $\dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$ and occurs at $\omega_r = \omega_n\sqrt{1-2\zeta^2}$
    - This can be obtained by differentiating the expression for the magnitude
    - For values of $\zeta > \dfrac{1}{\sqrt{2}}$, the resonant peak does not exist
    - The smaller ζ is, the closer the peak is to ω_n and the larger the magnitude of the peak
– Phase:
  * For $\omega \ll \omega_n$, $\angle 1 = 0°$
  * For $\omega \gg \omega_n$, $\angle (j\omega)^{\pm 2} = \pm 180°$
  * For $\omega \approx \omega_n$, $\angle(\pm j2\zeta) = \pm 90°$
  * The smaller the ζ, the faster the phase will transition between $0°$ and $\pm 180°$
    - For $\zeta = 0$, the transition is essentially a step function and the change is an instantaneous jump
    - For $\zeta = 1$, we just have a multiplication of two class 2 terms with the same break point

- Process for plotting a composite Bode plot:
  1. Manipulate the transfer function into Bode form to identify all break point frequencies
  2. Plot the low-frequency asymptote: Determine the value of n for the class 1 term and plot its magnitude as a line with slope of n passing through K_0 at $\omega = 1$
  3. Draw the asymptotes for the magnitude plot: Extend the low-frequency asymptote until the next break point, then change the slope by ± 1 or ± 2 depending on the class of the break point and whether it is numerator or denominator; repeat until all break points are accounted for
  4. Correct the magnitude values at break points:
     – For class 2, increase the magnitude by a factor of 1.41 (numerator) or decrease by a factor of 0.707 (denominator)
     – For class 3, change by a factor of (2ζ) (numerator) or a factor of $\dfrac{1}{2\zeta}$ (denominator)
     – Note these values may change of break points are close together; if break points are less than a factor of 10 away, the break point offsets are inaccurate
  5. Plot the low-frequency asymptote of the phase curve: $\phi = n \cdot 90°$
  6. Draw the horizontal asymptotes for phase: Change the value of the phase asymptote by $\pm 90°$ for class 2 break points and $\pm 180°$ for class 3 break points for each break point in ascending order
  7. Determine intermediate asymptotes for each break point

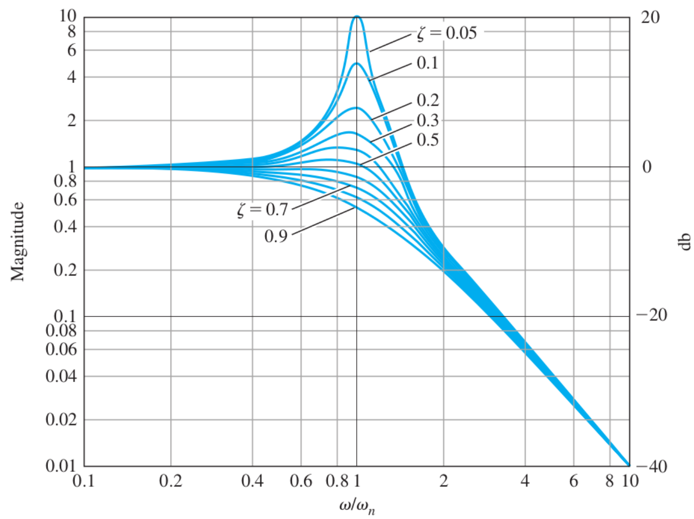

Figure 4: Magnitude plot of $\left(\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1\right)^{-1}$.

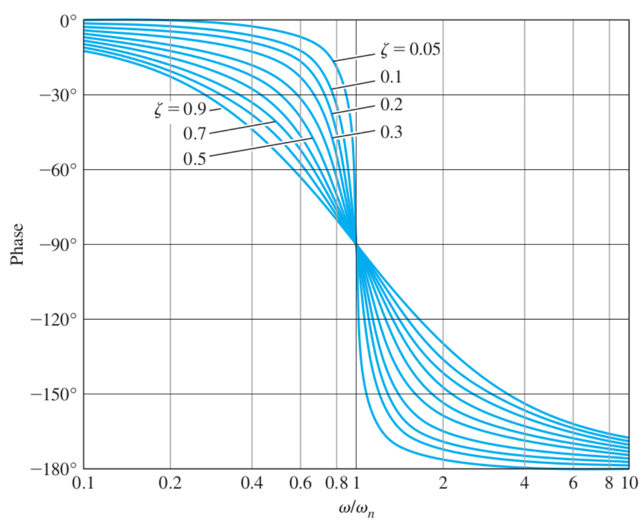

Figure 5: Phase plot of $\left(\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1\right)^{-1}$.

8. Add each phase curve together graphically

- Example: $G(s) = \dfrac{2000(s+0.5)}{s(s+10)(s+50)}$
  - $G(s) = 2s^{-1}\dfrac{\left(\frac{s}{0.5}+1\right)}{\left(\frac{s}{10}+1\right)\left(\frac{s}{50}+1\right)}$
  - Class 1: $2(j\omega)^{-1}$
  - Class 2: $\left(\dfrac{j\omega}{0.5}+1\right)$ with break point 0.5, $\left(\dfrac{j\omega}{10}+1\right)^{-1}$ with break point 10, $\left(\dfrac{j\omega}{50}+1\right)^{-1}$ with break point 50
  - Steps:
    1. Bode form: $2(j\omega)^{-1}\dfrac{\left(\frac{j\omega}{0.5}+1\right)}{\left(\frac{j\omega}{10}+1\right)\left(\frac{j\omega}{50}+1\right)}$
    2. From the class 1 term: At $\omega = 1$, the gain is 2; the slope is -1
    3. Continue the slope of -1 until the first break point 0.5, then increase slope by 1 (to 0); next break point is at 10, decrease slope by 1 (to -1); next break point is at 50, decrease slope by 1 (to -2)
    4. Increase magnitude by a factor of 1.41 at break point 0.5; decrease by a factor of 0.707 at break point 10; decrease by a factor of 0.707 at break point 50
    5. Low-frequency phase asymptote: $\phi = -90°$
    6. Increase phase by 90° at $\omega = 0.5$ (to 0°), decrease by 90° at 10 (to $-90°$), decrease by another 90° at 50 (to $-180°$)
    7. Draw the phase curves for the individual terms
    8. Graphically add the individual phase curves to obtain the final phase plot

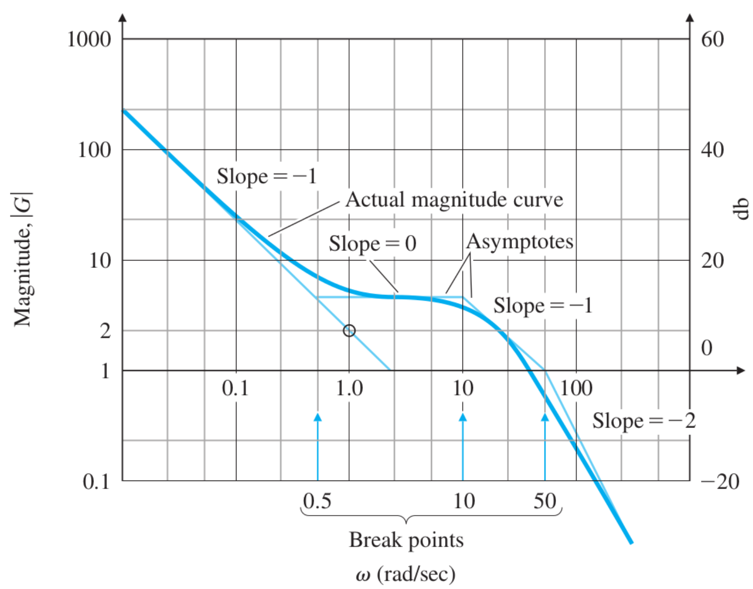

Figure 6: Magnitude plot of $G(s) = \dfrac{2000(s+0.5)}{s(s+10)(s+50)}$.

- Example: $G(s) = \dfrac{10}{s(s^2+0.4s+4)}$
  1. Bode form: $G(j\omega) = 2.5(j\omega)^{-1}\dfrac{1}{\left(\left(\frac{j\omega}{2}\right)^2 + 2(0.1)\left(\frac{j\omega}{2}\right)+1\right)}$
  2. Class 1 term: $2.5(j\omega)^{-1}$
     - First asymptote with slope of -1 having a value of 2.5 at $\omega = 1$
  3. Class 3 term: $\omega_n = 2$ and $\zeta = 0.1$, denominator

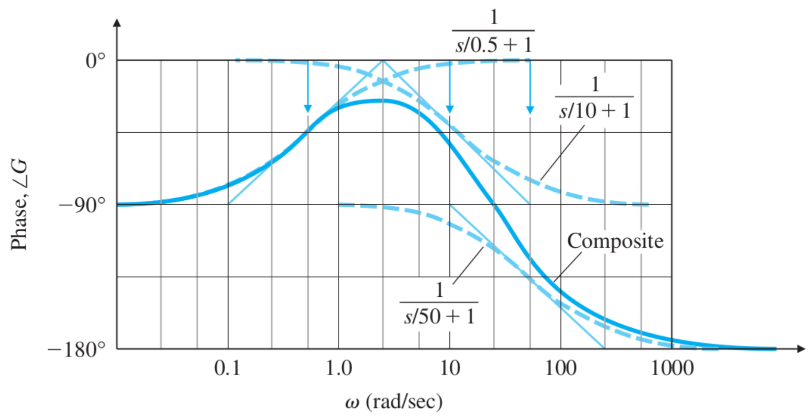

Figure 7: Phase plot of $G(s) = \dfrac{2000(s+0.5)}{s(s+10)(s+50)}$.

- Decrease the asymptote slope by 2 at $\omega = 2$

4. Increase magnitude by a factor of $\dfrac{1}{2(0.1)} = 5$ at the break point, and plot the magnitude
5. Low-frequency asymptote at $\phi = -90°$
6. Decrease phase by $-180°$ at $\omega = 2$ (to $-270°$)
7. Draw the phase plot

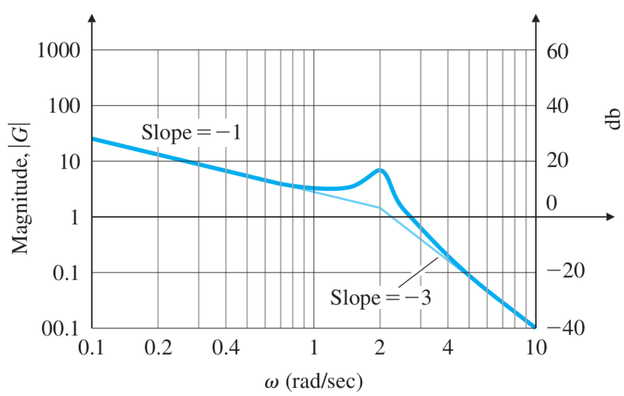

Figure 8: Magnitude plot of $G(s) = \dfrac{10}{s(s^2+0.4s+4)}$.

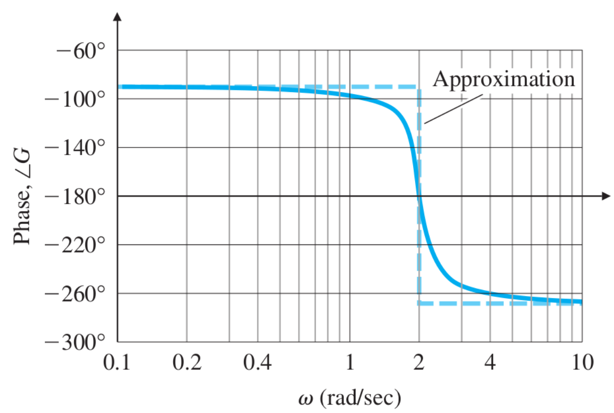

Figure 9: Phase plot of $G(s) = \dfrac{10}{s(s^2 + 0.4s + 4)}$.