Lecture 22, Mar 28, 2024

Frequency Response Design Method

- For a (stable) LTI system $G(s)$, the steady-state response to an input $u(t) = A\sin(\omega_0 t)1(t)$ is given by $y_{ss}(t) = A|G(j\omega_0)|\sin(\omega_0 t + \angle G(j\omega_0))$
  - The response is a sinusoid of the same frequency, scaled by a factor of $|G(j\omega_0)|$ (the magnitude of the transfer function, known as the *gain* or *amplitude/magnitude ratio*), with a phase shift of $\angle G(j\omega_0)$ (the phase of the transfer function)
  - Knowing the magnitude $M(\omega)$ and phase $\phi(\omega)$ of $G(j\omega)$ for all possible frequencies ω fully specifies the transfer function
- In general, the complete response is the sum of a number of exponentials and a sinusoid; since the system is stable, all the exponentials decay to 0 as $t \to \infty$ and we are only left with the sinusoid
- Example: RC circuit, output $y(t)$ is the voltage across the capacitor, input $\mathcal{K}u(t)$ is an input voltage that is sinusoidal
  - $RC\frac{\mathrm{d}y}{\mathrm{d}t} + y(t) + \mathcal{K}u(t) \implies \frac{\mathrm{d}y}{\mathrm{d}t} + ky(t) = u(t)$ where $k = \frac{1}{RC}$, assuming $\mathcal{K} = RC$
  - $G(s) = \frac{1}{s+k}$
  - Given $u(t) = \sin(10t)1(t)$, $U(s) = \frac{10}{s^2 + 100}$
  - At $s = j10$, $|G(j10)| = \frac{1}{\sqrt{1^2 + 10^2}}$ and $\angle G(j10) = -\tan^{-1}\left(\frac{10}{1}\right)$
  - Therefore the response is $y(t) = \frac{1}{\sqrt{101}}\sin(10t - \tan^{-1}(10))$
- Example: lead network $D_c(s) = K\frac{T_s + 1}{\alpha T_s + 1}$ for $\alpha < 1$
  - Note that this is mathematically identical to the form of the lead compensator we had before, but this form is more common and convenient for frequency response design
    * The zero is at $\frac{1}{T}$, the pole at $\frac{1}{\alpha T}$ and the gain is $\frac{K}{\alpha}$
  - Frequency response: $D_c(j\omega) = K\frac{T_j\omega + 1}{\alpha Tj\omega + 1}$
  - Gain: $M = |K|\frac{\sqrt{1 + \omega^2 T^2}}{\sqrt{1 + \alpha^2\omega^2 T^2}}$
  - Phase: $\phi = \tan^{-1}(\omega T) - \tan^{-1}(\alpha\omega T)$
  - For $\omega \to 0$, we have $M \to |K|$ and $\phi \to 0$
  - For $\omega \to \infty$ we have $M \to \left|\frac{K}{\alpha}\right|$ and $\phi \to 0$
- The gain and phase for a range of values of ω can be summarized in a *Bode plot*
  - The top plot is the magnitude plot; the bottom plot is the phase plot
  - The bode plot is log-log for magnitude and semi-log for phase
    * Using a log-log plot for gain allows us to cover a wide range of ω and gain, and also allows us to simply add up the magnitude plots of transfer functions to get the final plot, since multiplication of gains is just addition of logs
  - The vertical axis of the magnitude plot often uses decibels, $\mathrm{dB} = 20\log|G(j\omega)|$
- Note in MATLAB, `bode(sys, w)` gives `[mag, phase]`, which we can plot to get the Bode plot
  - Use `logspace()` to get the points for `w`

System Behaviour From Frequency Response

- The gain and phase of the system's frequency response completely determines the behaviour of the system; we design using it just like we design using the root locus
  - The root locus is to the root locus design method as the Bode plot is to the frequency design method

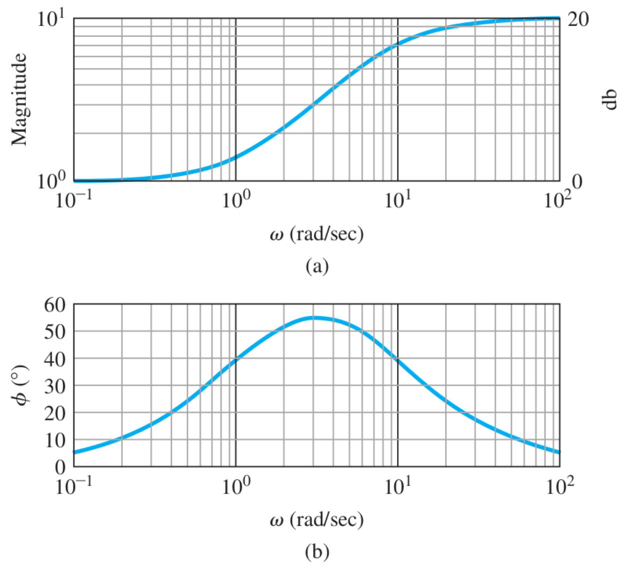

Figure 1: Bode magnitude and phase plots for the lead compensator, for $K = 1, \alpha = 0.1, T = 1$.

- Typical closed-loop systems exhibit a low-pass filter behaviour
  - The gain is close to 1 at lower frequencies, i.e. the output follows the input well
  - Beyond a certain frequency, the gain deviates from 1; for most systems, it increases first before decreasing
  - For most systems when the frequency gets very large the gain approaches 0, i.e. the output stops following the input at all
- The *bandwidth* ω_{BW} is defined as the highest frequency ω where the output still tracks the (sinusoidal) input in a satisfactory manner
  - Traditionally we define this to be when the gain hits $\sqrt{2}/2 = 0.707$
  - This is known as the *half-power point*; if the gain is a voltage gain, then at this point, the power of the response will be only half
  - A higher bandwidth means a faster response – the larger ω_{BW} is, the larger ω_n is and the shorter our rise and peak times
- The *resonant peak* M_r is the maximum value of the amplitude ratio
  - M_r has a direct relationship with ζ, so we can estimate the damping and overshoot of the system from M_r
- When we design a controller, we examine its bode plot and tune the gains to get the desired bandwidth and resonant peak, just like we identify pole locations on a root locus

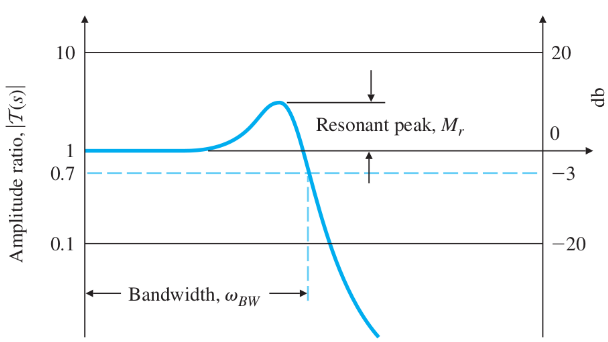

Figure 2: Definitions of bandwidth and resonant peak.