Lecture 21, Mar 25, 2024

Design for Dynamic Compensation (Continued)

Lead Compensator

- Example: $G(s) = \dfrac{1}{s(s+1)}$; design a lead compensator for the position control system to provide an overshoot of no more than 20% and rise time of no more than 0.3 seconds
  - This gives us a required damping ratio of $\zeta \geq 0.5$ and $\omega_n \geq 6\,\text{rad/s}$; we will choose $\omega_n \geq 7$ for some margin
  - Initial trial with $D_c(s) = K\dfrac{s+2}{s+10}$
    * We start with a zero at $s = -2$, since this is in the range of 1/4 to 1 times the natural frequency we want
    * Start with a pole of 10, at 5 times the location of the zero (recall rule of thumb was 5 to 25 times)
    * By drawing out the circle corresponding to $\omega_n = 7$ and the angle for $\zeta = 0.5$, we find a small segment on the root locus that gives the desired response
    * Note that the additional pole on the real axis is very close to a closed-loop zero (which are the same as the open-loop zeros due to unity feedback), so its effects are small
    * However, when we plot the response for $K = 70$ we see an overshoot of 22%
      - From here, we can try to lower K, but this is not the best option
      - We can increase the pole slightly, so the response is closer to that of a PD controller
      - We could also try to increase the zero, but we chose to increase the pole first since we have more range on it
  - Second trial with $D_c(s) = K\dfrac{s+2}{s+13}$
    * Now with a gain of $K = 91$ we have a controller with rise time of 0.19 seconds and overshoot of 17%

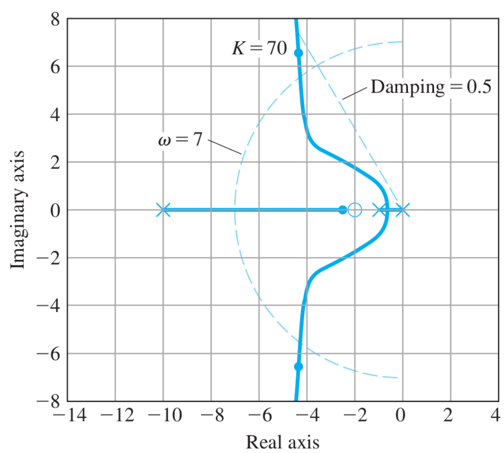

Figure 1: Root locus for $z = -2, p = -10$.

- In general when designing a closed-loop system, we typically start with a lead compensator:
  1. Determine where the closed-loop roots need to be to meet the desired physical response characteristics
  2. Create a root locus with only a proportional controller
  3. If more damping is needed, choose z to be 1/4 to 1 times the desired ω_n and pick p to be 5 to 25 times z
  4. If less damping is needed, decrease p; if more damping is needed, increase p and/or decrease z
     - The ratio p/z should be as low as possible (less than 25) in order to minimize the effects of

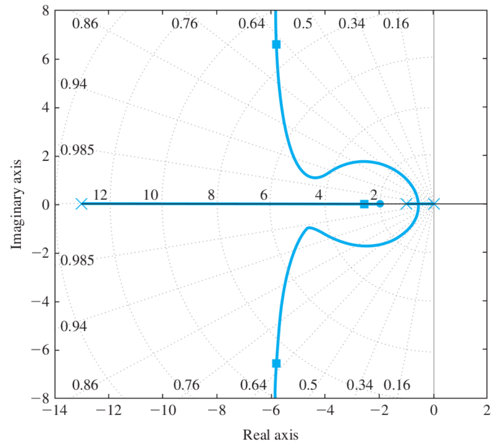

Figure 2: Root locus for $z = -2, p = -13$.

noise from a derivative controller

5. When values of z and p are found so that the root locus passes through the desired region, select the value of K and check the step response
6. Determine if the value of K meets the steady-state error requirements; if a value of K that meets the requirements cannot be found, add integral control or a lag compensator

- The lead compensator will make the steady-state error worse (for the same value of K)
  - The position constant is $K_p = \lim_{s\to 0} K\dfrac{s+z}{s+p}G(s) = K\dfrac{z}{p}\lim_{s\to 0} G(s)$
  - Since $p > z$, overall this makes K_p smaller, making e_{ss} larger
  - In order to reduce the steady-state error again, we want to introduce another term $\dfrac{s+z_2}{s+p_2}$ where $z_2 > p_2$, so the position constant is increased
    * This is the idea behind the lag compensator

Lag Compensator

- Lag compensation has a similar effect as an integrator in decreasing the steady-state error at low frequencies, without affecting the transient response created by the lead compensator
  - The position/velocity/acceleration constant is increased by a factor equal to z/p per the above discussion
  - The ratio z/p is typically between 3 to 10; anything more than this could affect the transient response
  - We choose the value of p and z to be extremely small (100-200 times smaller than the closed-loop ω_n), so $\dfrac{s+z}{s+p} \approx 1$ for any nonzero s, therefore it won't affect the transient response
  - Note that we need to be mindful of the resolution of our controller; if z and p are too small, it may not be practically implementable
- Example: Increase K_v for the previous system to decrease the steady-state error, without changing its transient response
  - Lag compensator $D_{c2}(s) = \dfrac{s+z}{s+p}$ where $z > p$
  - Uncompensated $K_v = \lim_{s\to 0} sD_{c1}(s)G(s) = 14$ so $e_{ss} = \dfrac{1}{14}$
  - Suppose we want to increase K_v to 70, so we need $\dfrac{z}{p} = \dfrac{70}{14} = 5$
  - Choose $z = 0.05, p = 0.01$
  - On the root locus, this adds a very small circle near the origin; the overall root locus is almost unchanged

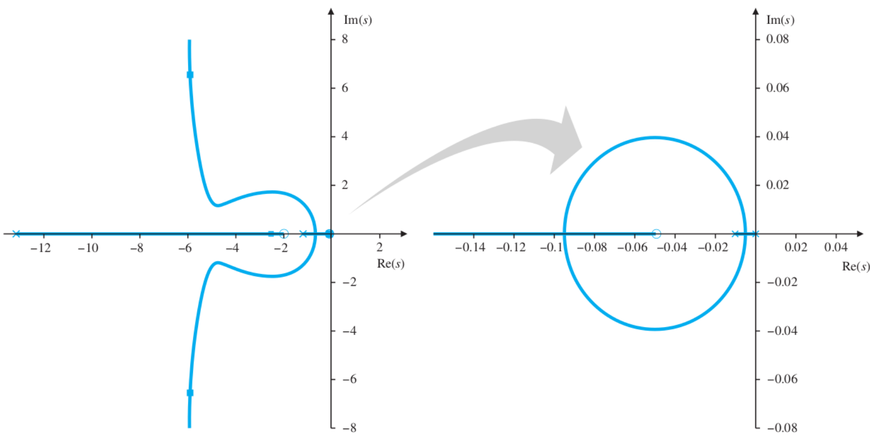

Figure 3: Root locus with lead and lag compensation.

- Lag compensator design process:
  1. Determine the amount of gain amplification we want to achieve the desired error constant, and determine the ratio z/p
  2. Select the value of z to be approximately 100 to 200 times smaller than the system's dominant natural frequency
  3. Plot the resulting root locus and verify that it is still satisfactory and adjust z and p as necessary
  4. Plot the step input to verify that the time domain response is still satisfactory
     - If the slow root of the lag compensator is too slow, increase z and p while keeping their ratio constant
     - Note that the closer z and p are to the dominant poles, the more effect they will have on the transient response

Notch Compensator

- A notch compensator is used to dampen the oscillation at some specific resonant frequency, e.g. due to a flexible mode in non-collocated control
  - The overall system response will have been handled by the other controllers; the notch compensator acts like a filter
  - Has form $\dfrac{s^2 + 2\zeta\omega_0 s + \omega_0^2}{(s+\omega_0)^2}$
  - The two real zeros cancel out the undesirable oscillatory poles in the system
  - The real poles are introduced so that the controller is causal and has a DC gain of 1, so the steady-state response is unaffected
  - Choose ω_0 to be very large as to not affect the transient response
- The position of the zero relative to the undesirable pole needs to be chosen to ensure that the resulting root locus is entirely in the LHP
  - Whether the zero should above or below the pole depends on the system
- Example: assume that the system has a flexible mode, so $G(s) = \dfrac{1}{s(s+1)} \cdot \dfrac{2500}{(s^2+s+2500)}$
  - The poles that were added are approximately $-0.5 \pm j50$; they are dominant and very lightly damped
  - Assume that we have the same lead-lag compensator from before
  - Add notch compensation $D_{c3}(s) = \dfrac{s^2 + 2\zeta\omega_0 s + \omega_0^2}{(s+\omega_0)^2} = \dfrac{s^2 + 0.8s + 3600}{(s+60)^2}$
    * The zeros are at approximately $-0.4 \pm j60$
      - Notice that zero is close to the pole but not exactly on it
      - The imaginary part is above the undesirable pole so that the root locus is entirely in the LHP

    - Typically the zero is chosen to be a little bit closer to the imaginary axis than the undesirable pole
  * The new poles we introduced at $s = -60$ are very far so they do not have any effect
- In practice, a notch compensator will often increase the overshoot, so we may need to iterate on the design
- Note practically, we design lead first, then notch, and finally lag, because the notch compensator affects the design of the lag compensator

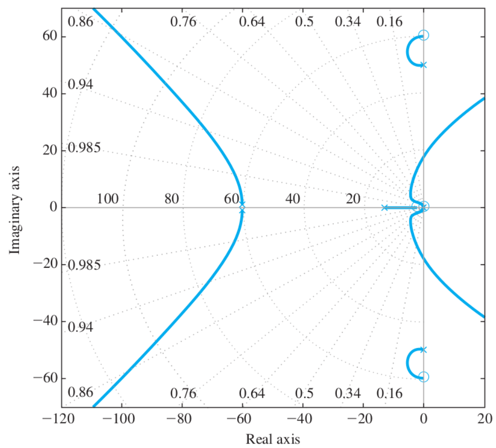

Figure 4: Root locus with flexible mode, lead, lag and notch compensation.

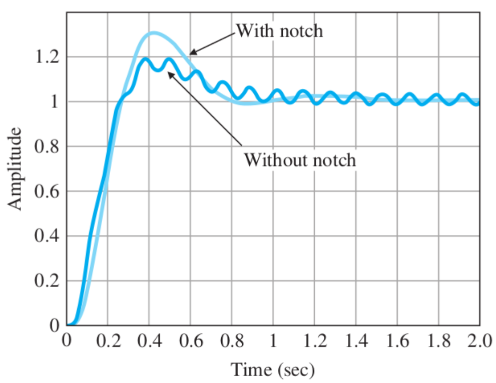

Figure 5: Step response with and without notch compensator.

Example: Quadrotor Drone Control (Pitch Axis)

- $G(s) = \dfrac{1}{s^2(s+2)}$
  - The double integrator represents a delay
- From the root locus we can see that with a proportional controller, the system is unstable for any value of K, since we have two branches going into the RHP
  - Adding a lead compensator $D_c(s) = K\dfrac{s+1}{s+10}$ pushes the root locus to the left, making the system stable
- Consider non-collocated behaviour where there is flexibility between the actuators and the body, so we introduce a flexible mode
- $G(s) = \dfrac{1}{s^2(s+5)} \cdot \dfrac{225}{((s+0.1)^2 + 15^2)}$
- Goal: $t_r \leq 1\,\text{s}, M_p \leq 40\%, t_s \leq 10\,\text{s}, K_a \geq 12\,\text{rad}$ without high frequency oscillations in response

- We can recognize that the system is type 2 due to the s^2 in the denominator, and adding compensators does not change the system type
- This translate to $\omega_n \geq 1.8\,\text{rad/s}, \zeta \geq 0.3, \sigma \geq 0.46$; we also need a lag and notch filter
- Proportional controller is again unstable with any gain
- Lead compensator: choose $z = 0.5$ (approximately $0.3\omega_n$), $p = 10$ (20 times the zero) so $D_{c1}(s) = \dfrac{s+0.5}{s+10}$
  - From the root locus we see that $K = 80$ is appropriate
  - Plotting the step response gives us a satisfactory overshoot and rise time
- Notch compensator: $D_{c3}(s) = \dfrac{(s+0.05)^2 + 16^2}{(s+16)^2}$
  - This cancels out the unwanted oscillations but slightly affects the transient response
  - Modify the lead compensator slightly to compensate
  - $K_a = \lim_{s\to 0} s^2 D_{c1}(s) D_{c3}(s) G(s) = 0.58$
- Lag compensator: need a ratio $\dfrac{z}{p} \geq \dfrac{12}{0.58} = 20.7$ so choose $D_{c2}(s) = \dfrac{s+0.02}{s+0.001}$
  - Modify the control gain as necessary