Lecture 19, Mar 18, 2024

Gain Selection from Root Locus

- Once we found a point on the root locus, s_0, that meets our requirements, we can find its value of K
- Since $L(s) = -\dfrac{1}{K}$ is the condition for the locus, $K = \dfrac{1}{|L(s)|} = \dfrac{|\prod_{i=1}^{n}(s_0 - p_i)|}{|\prod_{i=1}^{m}(s_0 - z_i)|} = \dfrac{\prod_{i=1}^{n}|s_0 - p_i|}{\prod_{i=1}^{m}|s_0 - z_i|}$
  - These magnitudes of the difference of s_0 from the poles and zeros can be obtained geometrically by measuring the distance of s from the roots and zeroes
- Once we have K, we can now solve for the values of s that make $L(s) = -\dfrac{1}{K}$ to find all the roots of the closed-loop system (since we only get one root initially)
- To identify $s_0 = -\sigma + j\omega$ given ζ:
  - We know $\dfrac{\omega}{\sigma} = \tan(\sin^{-1}\zeta)$
  - Substitute s_0 into $L(s) = -\dfrac{1}{K}$ and solve for the value of K
  - This will give us two equations, one for the real part (containing K), and another one for the imaginary part (which should equal 0)
  - Using the relation between σ and ω we can solve for their values using the imaginary equation
  - Substitute these values back into the real equation to solve for K

Example: 1-DoF Satellite Attitude Control

- Consider planar angular control of a satellite with a thruster generating a force F_c, and a disturbance M_D causing an unwanted moment
- $T_C + M_D = F_C d + M_D = I\ddot{\theta}$ where d is the distance from the centre of mass to the thruster and I is the satellite's moment of inertia
- Transfer function: assume $M_D = 0$, so $\dfrac{\Theta(s)}{T_C(s)} = G(s) = \dfrac{1}{Is^2} = \dfrac{A}{s^2}$
  - This a double-integrator
- Now consider an instrument attached to the satellite via a flexible boom, which can bend and vibrate
  - The total system has two degrees of freedom, the rotation of the satellite and the rotation of the instrument boom
  - The boom is modelled as a (rotational) spring-dashpot system between two discs
  - Bottom disc (attached to satellite): $T_C = I_1\ddot{\theta}_1 + b(\dot{\theta}_1 - \dot{\theta}_2) + k(\theta_1 - \theta_2)$
  - Top disc (attached to instrument): $0 = I_2\ddot{\theta}_2 + b(\dot{\theta}_1 - \dot{\theta}_1) + k(\theta_2 - \theta_1)$
  - We will simplify the system and assume $b = 0$
- Laplace transform:
  - $T_C = (I_1 s^2 + k)\Theta_1(s) - k\Theta_2(s)$
  - $0 = -k\Theta_1(s) + (I_2 s^2 + k)\Theta_2(s)$
- For this system, we can have two cases: either we want to control the attitude of the satellite, or the attitude of the instrument
  - $\dfrac{\Theta_1(s)}{T_C(s)} = \dfrac{I_2 s^2 + k}{I_1 I_2 s^2\left(s^2 + \frac{k}{I_1} + \frac{k}{I_2}\right)}$
    * Here we are controlling the side attached to the satellite
    * This is the case of *collocated control*: both the actuator and the sensor dynamics are on one body
  - $\dfrac{\Theta_2(s)}{T_C(s)} = \dfrac{k}{I_1 I_2 s^2\left(s^2 + \frac{k}{I_1} + \frac{k}{I_2}\right)}$
    * Here we are controlling the instrument boom
    * This is the *non-collocated* case: the actuator and sensor are not on the body we want to control
  - Notice that the collocated case has 2 zeros, which the non-collocated case misses – we will later see that the zeros in the first case make the control a lot simpler

- Consider a proportional controller $D_c(s) = k_P$ to control only the satellite without the boom, $\dfrac{\Theta}{T_C} = \dfrac{A}{s^2}$
  - Closed loop TF: $\dfrac{k_P \frac{1}{s^2}}{1 + k_P \frac{1}{s^2}}$ with characteristic equation $1 + k_P \dfrac{1}{s^2} = 0$
  - This is already in root locus form; $L(s) = \dfrac{1}{s^2} \implies b(s) = 1, a(s) = s^2$
  - Root locus determination:
    1. Two branches, both starting at $s = 0$, both going to infinity since there are no open-loop zeros
    2. No segments on the real axis; since both open-loop poles are at $s = 0$, for $s < 0$ we are on the left of 2 poles, and for $s > 0$ we are on the left of none
    3. Two asymptotes, intersecting at $\alpha = 0$ and at angles $\pm 90°$
    4. Branches have departure angles from $s = 0$ of $\pm 90°$ (one goes up, one goes down)
  - Notice that now all poles are on the imaginary axis – no matter what we do, we get oscillations with no damping

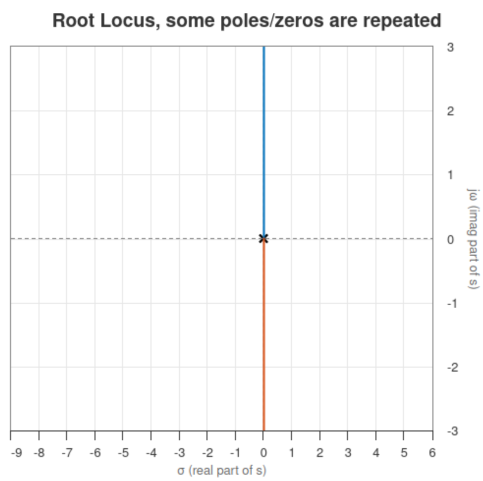

Figure 1: Root locus when using a proportional controller.

- Now consider using a PD controller $D_c(s) = k_P + k_D s$
  - Closed loop TF: $\dfrac{(k_P + k_D s)\frac{1}{s^2}}{1 + (k_P + k_D s)\frac{1}{s^2}}$ with characteristic equation $1 + (k_P + k_D s)\dfrac{1}{s^2} = 0$
  - Assume $k_D = K$ and $\dfrac{k_P}{k_D} = 1$, the characteristic equation is $1 + K\dfrac{s+1}{s^2} = 0$
    * The derivative gain introduced an open-loop zero to the system
  - Root locus:
    1. Two branches, both starting at $s = 0$, one of them going to the zero at $s = -1$, and the other going to ∞
    2. On the real axis, everywhere to the left of $s = -1$ is a part of the root locus, since that is to the left of 2 poles and 1 zero
    3. One asymptote along the negative real axis
    4. Departure angles from double pole at $s = 0$ are $\pm 90°$
    5. Two branches on the real axis meet at $\pm 90°$
    6. Break-in point at $s = -2$
  - Notice that the additional zeros has "pulled" the root locus to the left, adding damping and allowing us to have a response that does not oscillate forever

- However, in the real world any controller using a derivative gain is non-casual; implementing it in software will greatly amplify the noise in the system
  - To remedy this, we can try to add a denominator to the controller to make it casual

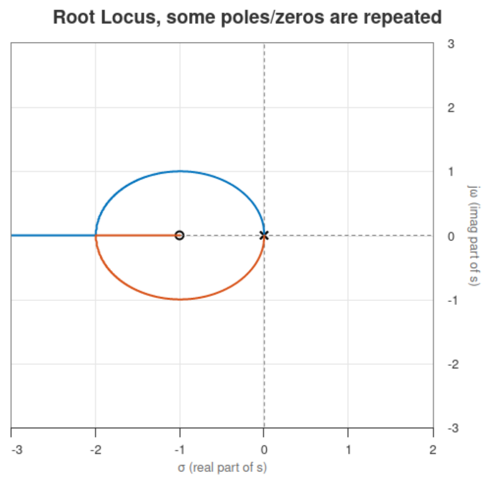

Figure 2: Root locus for the PD controller.

- We add a factor in the denominator of $\frac{s}{p} + 1$
  * If we choose p to be large, this will have little effect on the system response, but we can make the system causal and practically workable
  * We make the order of the denominator as small as possible to reduce sluggishness

- PD controller with *lead compensator*: $D_c(s) = k_P + \frac{k_D s}{\frac{s}{p} + 1}$
  - $D_c(s) = k_P + \frac{pk_P s}{s+p} = \frac{(k_P + pk_D)s + k_P p}{s+p} = \frac{(k_P + pk_D)\left(s + \frac{k_P p}{k_P + pk_D}\right)}{s+p}$
  - Let $k_P + pk_D = K$ and $\frac{k_P p}{k_P + pk_D}$ so $D_c(s) = K\frac{s+z}{s+p}$
    * With the large p, the pole it introduces is very far in the negative real axis, so it has a very small effect on the overall system
  - Characteristic equation: $1 + D_c(s)G(s) = 1 + K\frac{s+z}{s^2(s+p)} = 0$
  - Consider the following cases of p and z:
    * $z = 1$ and $p = 12$:
      - Root locus determination:
        1. 3 branches, two starting at $s = 0$, one starting at $s = -12$, one branch ends at $s = -1$, two at infinity
        2. Real axis $-12 \leq s \leq -1$ is on the locus
        3. 2 asymptotes centered at $-\frac{11}{2}$ at angles $\pm 90°$
        4. Departure angles at $s = 0$ are $\pm 90°$, at $s = -12$ is $0°$
        5. Break-in point at angle of $\pm 90°$
        6. Break-in point at $s = -2.3$ for the two branches starting at $s = 0$; two other branches depart at $s = -5.2$
      - We see that the root locus is close to that of just a PD controller
    * $z = 1$ and $p = 4$:
      - Now the root locus branches are pushed to the right, causing oscillatory responses
      - The pole being much closer means that it now starts to matter
    * $z = 1$ and $p = 9$:
      - For this in-between value we see that the new pole does impact the root locus, but the impact is smaller
    * As the pole gets closer to the zero, the branches begin to merge together
    * The pole should always be placed as far away as possible from the zero, but this has tradeoffs

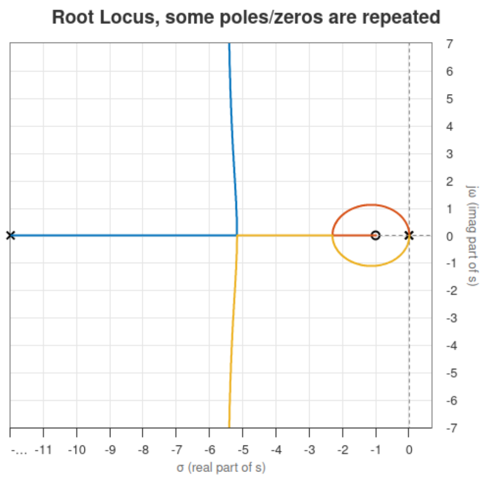

Figure 3: Root locus for the lead compensator, for $z = 1, p = 12$.

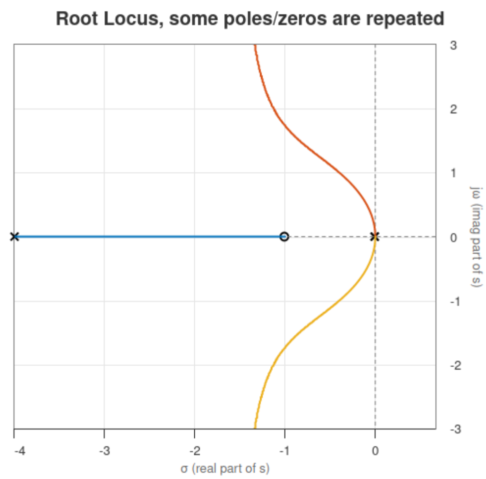

Figure 4: Root locus for the lead compensator, for $z = 1, p = 4$.

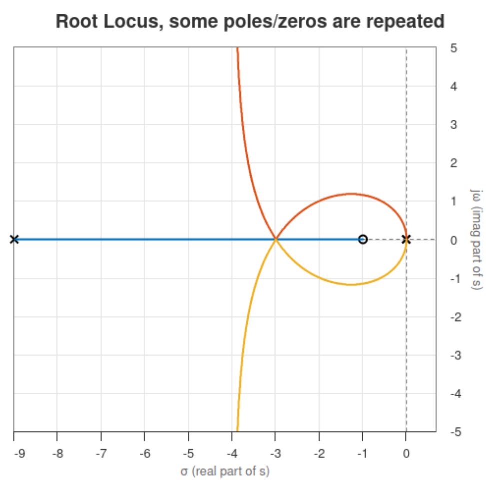

Figure 5: Root locus for the lead compensator, for $z = 1, p = 9$.