Lecture 17, Mar 11, 2024

Root-Locus Design Method

- A graphical method (set of rules) for finding the *locus* (set of locations on a line) of the roots of a system's characteristic equation, as a result of changing parameters
  - Allows us to find how the roots of a system move as a result of variation in some system parameter
  - e.g. we can find how the poles move as a result of changing the gain, so we can assess the system's stability, speed, etc
  - The *root locus* is the set of all locations that a root can take as a result of changing some parameter
  - Note the parameter must affect the characteristic equation linearly
- In controls, we use this to find how the roots of the characteristic equation (i.e. the poles) are affected by changing system gains
- Consider the closed-loop transfer function $\frac{Y(s)}{R(s)} = \mathcal{T}(s) = \frac{D_c(s)G(s)}{1 + D_c(s)G(s)H(s)}$
  - Rewrite the characteristic equation into the form of $1 + D_c(s)G(s)H(s) = a(s) + Kb(s) = 0$
  - Then we have $1 + K\frac{b(s)}{a(s)} = 0 \implies 1 + KL(s) = 0$ where $L(s) = \frac{b(s)}{a(s)} = -1\frac{1}{K}$
    - Writing $L(s) = \frac{b(s)}{a(s)}$ is known as the root-locus or *Evans form*
  - Now our poles are locations where $L(s) = -\frac{1}{K}$, which is often a negative real number
  - Since the original poles are at $D_c(s)G(s)H(s) = -1$, $KL(s) = D_c(s)G(s)H(s)$, the open-loop transfer function
    - Sometimes we will just refer to the open-loop transfer function as $L(s)$ and ignore the K
  - Most often K is a positive real number since it is a gain, but in rare cases we can also deal with $K < 0$
- The roots of the characteristic equation are located where the open-loop transfer function of the system becomes a real negative value
  - Therefore we can plot the location of all possible roots s of the characteristic equation by varying K; this is the root locus
  - The root locus allows us to select the best controller gains and study the effect of potentially adding additional poles and zeros
- Let $b(s) = s^m + b_1 s^{m-1} + \cdots + b_m = \prod_{i=1}^{m}(s - z_i), a(s) = s^n + a_1 s^{n-1} + \cdots + a_n = \prod_{i=1}^{n}(s - p_i)$
  - z_i are the open-loop zeroes, p_i are the open-loop poles
  - Note $n \geq m$ because $L(s) \propto D_c(s)G(s)H(s)$ is causal
- Let $a(s) + Kb(s) = \prod_{i=1}^{n}(s - r_i)$ (note $n \geq m$ so the summation ends at n)
  - The r_i are the closed-loop poles; note this is not the same as the open-loop poles
  - Our goal is to draw all the possible locations of r_i for different values of K
- Example: $D_c(s) = K$, $G(s) = \frac{1}{s(s+c)}$ and consider $c = 1$; plot the root locus with respect to K
  - $G(s)D_c(s) = \frac{K}{s^2 + s} \implies \mathcal{T}(s) = \frac{K}{(s^2 + s) + K} = \frac{K}{a(s) + Kb(s)}$
  - We have $b(s) = 1, a(s) = s^2 + s \implies m = 0, n = 2, z_i = \varnothing, p_i = \{0, -1\}, r_i = -\frac{1}{2} \pm \frac{\sqrt{1 - 4K}}{2}$
  - $L(s) = \frac{b(s)}{a(s)} = \frac{1}{s(s+1)}$
  - For $K = 0$, we have two real roots $r_1 = -1, r_2 = 0$
    - Notice that these are the same as the open-loop poles, since $K = 0 \implies a(s) + Kb(s) = a(s)$
  - For $K = \frac{1}{4}, r_1 = r_2 = -\frac{1}{2}$
  - For $K > \frac{1}{4}$ the roots will be imaginary, and the pair of poles will move up further from the real

axis

- The two directions that the poles move in are the 2 *branches*
    * The branches start at the open loop poles, which are the *start points*
    * The locus has one *breakaway point*, where the two poles join and separate
        - Note breakaway points are when the poles move in from the real axis
- Suppose we want $\zeta = 0.5$, geometrically we can draw out a line at an angle $\sin^{-1}\zeta = 30°$ from the origin, and find where it intersects with the root locus

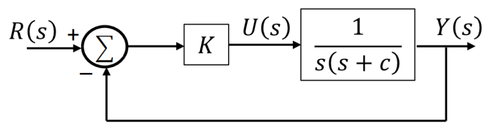

Figure 1: Example feedback control system.

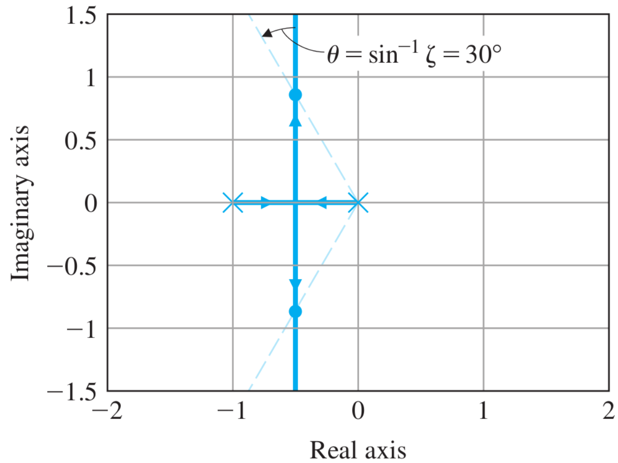

Figure 2: Root locus plot of the example system.

- Example: root locus of the previous system with respect to c
    - $\mathcal{T}(s) = \dfrac{1}{s^2 + 1 + cs} = \dfrac{1}{a(s) + cb(s)} \implies \begin{cases} b(s) = s \\ a(s) = s^2 + 1 \end{cases}, L(s) = \dfrac{s}{s^2+1}$
    - The roots are $z_i = 0, p_i = \pm j$
    - $a(s) + cb(s) = s^2 + cs + 1 = 0 \implies r_1, r_2 = -\dfrac{c}{2} \pm \dfrac{\sqrt{c^2 - 4}}{2}$
    - For $c = 0$ we have $r_1, r_2 = \pm j$, giving the start of the plot
    - For $c = 2$ the two roots meet at $r_1 = r_2 = -1$
    - As $c \to \infty$, one of the poles moves to $-\infty$ while the other converges to 0
    - The circle on the diagram indicates the location of $z_1 = 0$
    - This root locus has 2 start points, 2 branches, and 1 *break-in point* (where the poles meet and separate, but they come from the imaginary axis)

Root Locus Determination

Definition

A *root locus* is the set of all possible values of s for which the characteristic equation $1 + KL(s) = 0$ holds, as the real parameter K varies from 0 to ∞ (sometimes $-\infty$). In controls, the characteristic equation is typically for a closed-loop system, so the roots of the locus are the system poles.

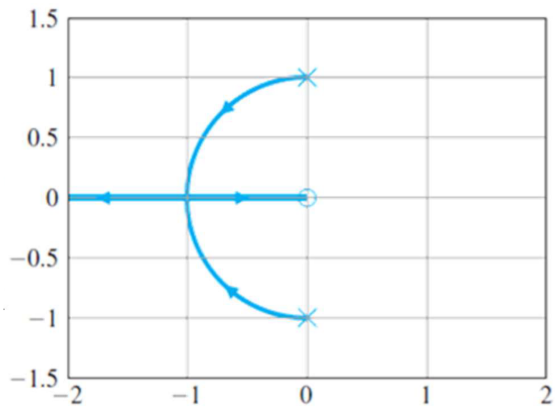

Figure 3: Root locus plot with respect to c for the example system.

- If K is real and positive, then $L(s)$ must be real and negative, so its phase must be $+180°$ (*positive locus*)
  - In rare cases K is negative, then $L(s)$ has a phase of 0 (*negative locus*)
- We can alternatively define the root locus as the set of points in the s-plane where the phase of $L(s)$ equals 180° for positive loci, or 0° for negative loci
  - This will help us plot the locus
- Recall that for $L(s) = \dfrac{b(s)}{a(s)}$ the phase of $L(s)$ is equal to the phase of $b(s)$ minus the phase of $a(s)$
- Consider a test point s_0
  - To find the phase of $L(s_0)$, we need to find the phase of $b(s_0)$ and $a(s_0)$
  - $\angle b(s) = \sum_{i=1}^{m} \angle(s_0 - z_i)$ and $\angle a(s) = \sum_{i=1}^{n} \angle(s_0 - p_i)$
  - We need to check that $\sum_{i=1}^{m} \angle(s_0 - z_i) - \sum_{i=1}^{n} \angle(s_0 - p_i) = 180° + 360°(l-1)$
  - The phase of each $s_0 - z_i$ is the angle from each open-loop zero to s_0; the phase of each $s_0 - p_i$ is the angle from each open-loop pole to s_0
  - Therefore we take the sum of the angles of s_0 from the open-loop zeros, denoted ϕ_i, and subtract the sums of the angles of s_0 from the open-loop poles, denoted ϕ_i

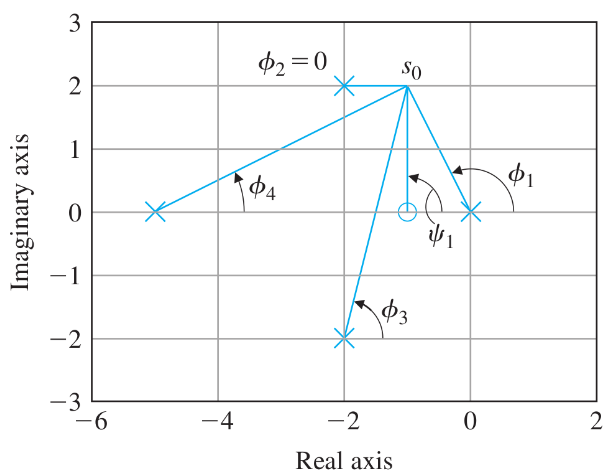

Figure 4: Testing whether a point s_0 is part of the root locus.