Lecture 15, Mar 4, 2024

PID Controllers

- For the following analyses we will assume unity feedback, but this is easily extended to other kinds of feedback

Proportional Control (P)

- The simplest controller simply applies a gain to the error feedback: $u(t) = k_P e_a(t) \implies D_{cl} = k_P$
- Consider a second order plant $G(s) = \dfrac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \implies GD_{cl}(s) = \dfrac{k_P K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
  - No poles at the origin in the open-loop transfer function, therefore this is a type 0 system
- Closed loop transfer function: $\dfrac{Y(s)}{R(s)} = \dfrac{GD_{cl}(s)}{1 + GD_{cl}(s)} = \dfrac{k_P K\omega_n^2}{s^2 + 2\zeta\omega_n s + (1 + k_P K)\omega_n^2}$
  - Notice that the new natural frequency is $\omega_n' = \sqrt{1 + k_P K\omega_n^2}$, which is increased
  - The new damping ratio is $\zeta' = \dfrac{\zeta}{\sqrt{1 + k_P K}}$, which is decreased (obtained by comparing $2\zeta'\omega_n'$ with $2\zeta\omega_n$)
  - Increased natural frequency leads to shorter rise time (faster system), but decreased damping leads to more overshoot
- For $R(s) = \dfrac{1}{s}$, we have $e_{ss} = \lim_{s \to 0} s\dfrac{1}{1 + GD_{cl}}\dfrac{1}{s} = \dfrac{1}{1 + k_P K}$
  - The steady-state error in the step response is reduced, but not eliminated entirely
- The same analysis can be made for the disturbance regulation

Integral Control (I)

- The integral controller applies $u(t) = k_I \displaystyle\int_0^t e_a(\tau)\,\mathrm{d}\tau$
  - Instead of the error itself, the control signal is proportional to the area underneath the error curve
- The controller transfer function is $\dfrac{U(s)}{E_a(s)} = D_{cl}(s) = \dfrac{k_I}{s}$
- Consider the same second-order plant $G(s) = \dfrac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$:
- Closed loop transfer function: $\dfrac{Y(s)}{R(s)} = \dfrac{k_I G(s)}{s + k_I G(s)} = \dfrac{k_I K\omega_n^2}{s^3 + 2\zeta\omega_n s^2 + \omega_n^2 s + k_I K\omega_n^2} = \mathcal{T}(s)$
  - Notice that this system is now third order; the increased order of the system makes it more sluggish, so the rise time increases and overshoot decreases
  - Taking $s \to 0$ we see that the DC gain is 1, so there is no more steady-state error
  - Unlike the second-order system, we can no longer conclude that the system is always stable, since this is a third-order system
  - Using the Routh criterion, we find that $k_I < \dfrac{2\zeta\omega_n}{K}$ is the maximum value of k_I for the system to be stable
    - The integral controller can destabilize the system!
  - The removal of steady-state error is a robust property, holding regardless of the value of k_I and plant parameters
    - We can find the sensitivity transfer function and show that this always goes to 0 as $s \to 0$
- $GD_{cl}(s) = \dfrac{k_I K\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$; $E_{cl}(s) = \dfrac{1}{1 + GD_{cl}}R(s)$
  - There is one pole at the origin, so this is a type 1 system
  - This can now follow a position setpoint with no error, and a velocity setpoint with constant error
  - Note we only get zero error for position setpoints when we have unity feedback!
  - The velocity constant is $K_v = \lim_{s \to 0} sGD_{cl}(s) = k_I K \implies e_{ss} = \dfrac{1}{k_I K}$

Derivative Control (D)

- The derivative controller applies $u(t) = k_D \dot{e}_a(t) \implies \dfrac{U(s)}{E_a(s)} = D_{cl}(s) = k_D s$
  - Derivative control tends to speed up the system, since it anticipates future behaviour of the system
- The closed-loop transfer function is $\dfrac{k_D K \omega_n^2 s}{s^2 + (2\zeta + k_D K \omega_n)\omega_n s + \omega_n^2}$ for the second-order plant
  - The additional zero speeds up the system and increases the overshoot
  - However, the damping ratio is increased to $\zeta' = \zeta + \dfrac{1}{2} k_D K \omega_n$, which decreases the overshoot
  - Overall, the combination leads to increased system speed and decreased overshoot
  - Furthermore, increased damping ratio and constant natural frequency moves the poles away from the imaginary axis, enhancing stability
- $GD_{cl}(s) = \dfrac{k_D K \omega_n^2 s}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
  - No poles at zero, therefore the system is type 0 and maintains constant error for a step input
  - The position constant is $K_p = \lim_{s \to 0} GD_{cl}(s) = 0$
  - The steady state error is $e_{ss} = \dfrac{1}{1 + K_p} = 1$
    - This means that the output ultimately converges to zero, i.e. the derivative controller can't do anything about the steady state error
- Generally, derivative control enhances the transient behaviour of the system but does nothing to its long-term behaviour
- The transfer function for the controller is not causal
  - This means we can't implement it with analog controllers
  - We can still implement this digitally, but taking numerical derivatives highly amplifies noise
  - Therefore, in reality derivative controllers may not be practical
  - Practically, we use another technique called lead functions instead of derivatives
- Derivative control can be used to damp the control response, so we don't get sharp reactions to suddenly changing signals
  - If there is a sudden jump in the output due to transient effects, there will be a jump in error and also $u(t)$, which is not desirable
  - A derivative feedback path will correct for this

Proportional-Integral Control (PI)

- $u(t) = k_P e_a(t) + k_I \int_0^t e_a(\tau)\,\mathrm{d}\tau \implies D_{cl}(s) = k_P + \dfrac{k_I}{s}$
- The closed-loop transfer function is $\dfrac{(k_P s + k_I) K \omega_n^2}{s^3 + 2\zeta\omega_n s^2 + (1 + k_P K)\omega_n^2 s + k_I K \omega_n^2}$
  - We still increase the system order, but also added a zero, which counteracts the slowdown effect
    - The final system can be faster than the initial plant
  - There is a zero that we can use to cancel a stable pole, which would make the system behave like second-order, making it easier to analyze and control
- $GD_{cl}(s) = \dfrac{(k_P s + k_I) K \omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$
  - The system is type 1, with $K_v = k_I K$ and steady-state error $e_{ss} = \dfrac{1}{k_I K}$
  - Also type 1 in regulation
- Stability criterion: $k_I < \dfrac{2\zeta\omega_n(1 + k_P K)}{K}$
- Note that we only have two adjustable parameters k_P and k_I, but there are 3 roots, so our ability to control the characteristic equation is limit
- Generally used to allow for a faster response compared to a pure integral controller

Proportional-Derivative-Integral Control (PID)

- $u(t) = k_P e_a(t) + k_I \int_0^t e_a(\tau)\,\mathrm{d}\tau + k_D \dot{e}_a(t) \implies D_{cl}(s) = k_P + \frac{k_I}{s} + k_D s$
- Second-order closed loop: $\dfrac{(k_D s^2 + k_P s + k_I)K\omega_n^2}{s^3 + (2\zeta + k_D K\omega_n)\omega_n s^2 + (1 + k_P K)\omega_n^2 s + k_I K\omega_n^2}$
  - We can fully control the location of the poles since there are 3 poles and we have 3 parameters
- $GD_{cl}(s) = \dfrac{(k_D s^2 + k_P s + k_I)K\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$
  - The system is type 1, and has $k_v = k_I K \implies e_{ss} = \dfrac{1}{k_I K}$
- Stability criterion: $k_I < \dfrac{(2\zeta + k_D K\omega_n)(1 + k_P K)\omega_n}{K}$

Example System: DC Servo Motor

- Consider the DC motor system derived earlier
- $l_a \dfrac{\mathrm{d}i_a}{\mathrm{d}t} + r_a i_a = v_a - K_e\dot{\theta}_m \implies (l_a s + r_a)I_a(s) = V_a(s) - K_e s\theta_m(s)$
  - This models the back EMF and inductive/resistive effects of the motor coil
- $J_m\ddot{\theta}_m + b_m\dot{\theta}_m = K_t i_a - \eta T \implies (J_m s + b_m)s\Theta_m(s) = K_t I_a(s) - \eta T(s)$
  - This models torque on the shaft, including friction and an external resisting force
- $V_a(s)$ is the input to the system, $T(s)$ is a disturbance, and $\Theta_m(s)$ is the final output
  - $V_a(s)$ first passes through a transfer function to get $I_a(s)$, then this is multiplied by K_t to get a torque
  - This is summed with the torque from the disturbance and passes through the mechanical transfer function to get $\dot{\theta}_m$
  - A final integrator gets us $\Theta_m(s)$
  - The back EMF introduces a feedback path with constant gain K_e

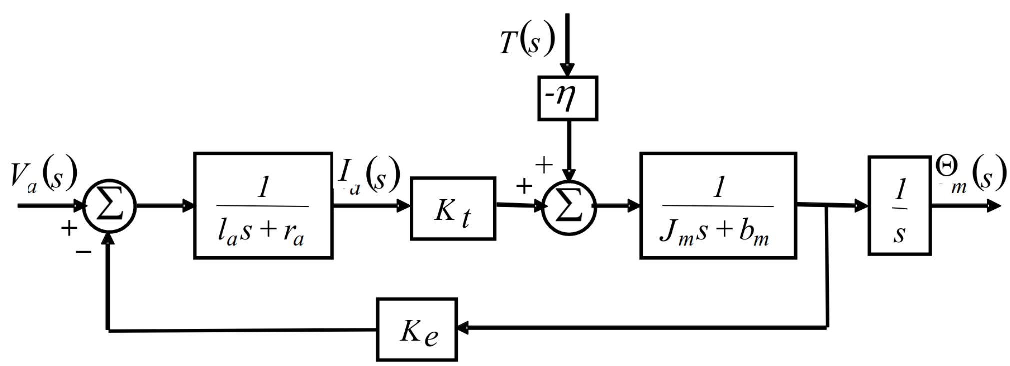

Figure 1: Block diagram for the DC motor system.

- The system has two inputs (V_a and T), since it is linear we can consider one at a time to get transfer functions
  - $\dfrac{\Theta_m(s)}{V_a(s)} = \dfrac{K_t}{s((l_a s + r_a)(J_m s + b_m) + K_c K_t)}$
  - $\dfrac{\Theta_m(s)}{T(s)} = \dfrac{-\eta(l_a s + r_a)}{s((l_a s + r_a)(J_m s + b_m) + K_e K_t)}$
- We can again make the simplifying assumption that the electrical part of the system operates on a much faster time scale than the mechanical part, so the inductance l_a can be taken to 0
  - $\dfrac{\Theta_m(s)}{V_a(s)} = \dfrac{\frac{K_t}{r_a}}{s\left(J_m s + \left(b_m + \frac{K_e K_t}{r_a}\right)\right)} = \dfrac{K}{s(\tau s + 1)}$

- $\frac{\Theta_m(s)}{T(s)} = \frac{-\eta}{s\left(J_m s + \left(b_m + \frac{K_e K_t}{r_a}\right)\right)} = \frac{C}{s(\tau s+1)}$
- $\tau = \frac{J_m r_a}{b_m r_a + K_e K_t}$
- $K = \frac{K_t}{b_m r_a + K_e K_t}$
- $C = \frac{-\eta r_a}{b_m r_a + K_e K_t}$
- We can now build a much simpler block diagram

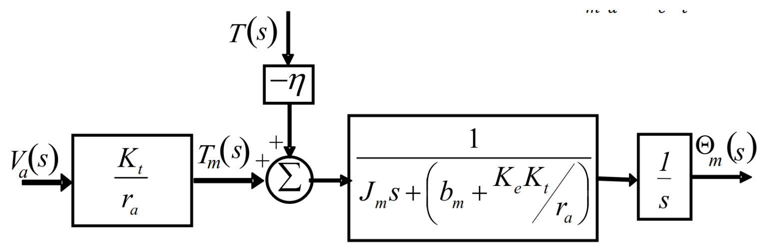

Figure 2: Simplified block diagram for the DC motor system.

- Now we close the loop, with a feedback transfer function $H(s) = hs$ and controller $D_{cl}(s)$
  - We can do this for either a position or velocity controller
  - Since we have non-unity feedback, we can no longer only look at the poles to tell the system type and must use brute force
- For position control: $\frac{\Theta_m(s)}{\Theta_{mr}(s)} = \frac{KD_{cl}(s)}{s(\tau s+1) + KhD_c(s)}$, $\frac{\Theta_m(s)}{T(s)} = \frac{-K\eta}{s(\tau s+1)+KhD_c(s)}$
  - Consider a PID controller: $D_{cl}(s) = k_P + \frac{k_I}{s} + k_D s$
  - $\mathcal{T}(s) = \frac{\Theta_m(s)}{\Theta_{mr}(s)} = \frac{Kk_P + \frac{Kk_I}{s} + Kk_D s}{\tau s^2 + s + Khk_P + \frac{Khk_I}{s} + Khk_D s} = \frac{\frac{1}{\tau}(Kk_D s^2 + Kk_P s + Kk_I)}{s^3 + \frac{1}{\tau}(Khk_D+1)s^2 + \frac{1}{\tau}Khk_P s + \frac{1}{\tau}Khk_I}$
  - $T_w(s) = \frac{\Theta_m(s)}{T_s(s)} = \frac{-K\eta}{\tau s^2 + s + Khk_P + \frac{Khk_I}{s} + Khk_D s} = \frac{-\frac{1}{\tau}K\eta s}{s^3 + \frac{1}{\tau}(Khk_D+1)s^2 + \frac{1}{\tau}Khk_P s + \frac{1}{\tau}Khk_I}$
- For tracking: $E(s) = \Theta_{mr}(s) - \Theta_m(s)$

$$= (1-\mathcal{T}(s))\Theta_{mr}(s)$$

$$= \left(\frac{s^3 \frac{1}{\tau}(Khk_D+1)s^2 + \frac{1}{\tau}Khk_P s + \frac{1}{\tau}Khk_I - \frac{1}{\tau}Kk_D s^2 - \frac{1}{\tau}Kk_P s - \frac{1}{\tau}Kk_I}{s^3 + \frac{1}{\tau}(Khk_D+1)s^2 + \frac{1}{\tau}Khk_P s + \frac{1}{\tau}Khk_I}\right)\Theta_{mr}(s)$$

  - For a step $\Theta_{mr}(s) = \frac{1}{s}$, so $e_{ss} = \lim_{s\to 0} sE(s) = \frac{\frac{1}{\tau}Kk_I(h-1)}{\frac{1}{\tau}Khk_I} = \frac{h-1}{h}$
  - We have a constant error, so this system is only type 0
  - Even though we have an integral term, the error was not reduced to 0 because the system is not unity feedback
    * If the system was unity feedback, then $h = 0$ and we would have a type 1 system
- For regulation: $E(s) = \Theta_{mr}(s) - \Theta_m(s)$

$$= -\Theta_m(s)$$

$$= -T_w T(s)$$

  - For a step disturbance $T(s) = \frac{1}{s}$
    * $e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} -T_w(s) = 0$
    * This system is higher than type 0
  - For a ramp disturbance $T(s) = \frac{1}{s^2}$

    * $e_{ss} = \lim_{s \to 0} -sT_w(s) = \frac{\frac{1}{\tau}K\eta}{\frac{1}{\tau}Khk_I} = \frac{\eta}{hk_I}$
    * Therefore this system is type 1 with respect to regulation
    * The velocity constant is $\frac{1}{e_{ss}} = \frac{hK_I}{\eta}$
- In summary, the system is type 0 with respect to tracking and type 1 with respect to regulation for PID
  - For PI, we have the same type 0 in tracking and type 1 in regulation
  - For P, we have type 0 in both regulation and tracking
- The same analysis can be applied for velocity control, where our feedback will be taken from $\Omega_m(s)$, the speed of the shaft
  - Construct the same transfer functions for regulation and tracking
  - For velocity control, we also have the same types with the controllers
- In general, for PI and PID control the system type is usually the same