Lecture 14, Feb 29, 2024

Control System Type

- The reference input $R(s)$ can often be approximated by a time domain polynomial $r(t) = Ct^k 1(t)$
  - e.g. for position $k = 0$, for velocity $k = 1$ and for acceleration $k = 2$
- The *type* of a closed-loop controller is the maximum order of the polynomial that the controller can follow with a constant error
  - e.g. if the system can follow a ramp with constant error, then it is a type 1 system
  - Any inputs of a higher order will lead to increasing error
  - Any inputs of a lower order will lead to zero error
- For unity feedback (i.e. $H(s) = 1$ or perfect sensors) and no disturbance ($W = V = 0$), the type of a system depends on the number of poles that its open loop transfer function, $HGD_{cl} = GD_{cl}$, has at the origin
  - $E_{cl}(s) = R(s) - Y(s) = \frac{1}{1 + GD_{cl}} R = \mathcal{S}(s)R(s)$
  - Let the reference input $r(t) = \frac{1}{k!} t^k 1(s) \implies R(s) = \frac{1}{s^{k+1}}$
  - $e_{ss} = \lim\limits_{t\to\infty} e(t) = \lim\limits_{s\to 0} sE_{cl}(s) = \lim\limits_{s\to 0} s \frac{1}{1 + GD_{cl}} \frac{1}{s^{k+1}}$
  - First consider if GD_{cl} has no pole at the origin
    * With $k = 0$, $e_{ss} = \lim\limits_{s\to 0} s \frac{1}{1 + GD_{cl}} \frac{1}{s} = \frac{1}{1 + GD_{cl}(0)} = \frac{1}{1 + K_0}$
      - Therefore for a step input we get a constant steady state error
    * For $k > 0$, $e_{ss} = \lim\limits_{s\to 0} s \frac{1}{1 + GD_{cl}} \frac{1}{s^{k+1}} = \frac{1}{1 + GD_{cl}(0)} \lim\limits_{s\to 0} \frac{1}{s^k} = \infty$
      - For any higher degree input, the error goes to infinity
  - Now consider $GD_{cl}(s) = \frac{\overline{GD}_c(s)}{s^n}$
    * $\overline{GD}_c(s)$ contains all terms of $GD_{cl}(s)$ except for poles at the origin, so $K_n = \overline{GD}_c(0)$ is a finite value
    * For $n = k = 0$ (type 0) we've shown above that $e_{ss} \to 0$
    * For $n = k \neq 0$, $e_{ss} = \lim\limits_{s\to 0} s \frac{1}{1 + \frac{\overline{GD_{cl}}(s)}{s^n}} \frac{1}{s^{k+1}} = \lim\limits_{s\to 0} \frac{s^n}{s^k(s^n + \overline{GD_{cl}}(s))} = \frac{1}{\overline{GD_{cl}}(0)} = \frac{1}{K_n}$
    * For $n > k$, $e_{ss} = \lim\limits_{s\to 0} \frac{s^n}{s^k(s^n + \overline{GD_{cl}}(s))} = \frac{1}{\overline{GD_{cl}}(0)} \lim\limits_{s\to 0} s^{n-k} = 0$
    * For $n < k$, $e_{ss} = \lim\limits_{s\to 0} \frac{s^n}{s^k(s^n + \overline{GD_{cl}}(s))} = \frac{1}{\overline{GD_{cl}}(0)} \lim\limits_{s\to 0} \frac{1}{s^{k-n}} = \infty$
- The type of a system is a *robust property*, i.e. it is independent of the parameters of the system
- For a type 0 system, we can define a *position constant*, $K_p = K_0 = \lim\limits_{s\to 0} GD_{cl}(s)$, so $e_{ss} = \frac{1}{1 + K_0}$ (known as the *position error constant*)
  - Note that this is the only one where the error constant is not a simple reciprocal
- For a type 1 system, we can define a *velocity constant*, $K_v = K_1 = \lim\limits_{s\to 0} sGD_{cl}(s)$, so $e_{ss} = \frac{1}{K_1}$
- For a type 2 system, we can define an *acceleration constant*, $K_a = K_2 = \lim\limits_{s\to 0} s^2 GD_{cl}(s)$, so $e_{ss} = \frac{1}{K_2}$
- Example: plant $G(s) = \frac{A}{\tau s + 1}$ with controller $D_{cl}(s) = k_P + \frac{k_I}{s}$
  - $GD_{cl}(s) = \frac{A(k_P s + k_I)}{s(\tau s + 1)}$ so this is a type 1 system
  - The velocity constant is $K_v = \lim\limits_{s\to 0} sGD_{cl}(s) = Ak_I$ so the steady-state error is $\frac{1}{Ak_I}$
- For non-unity feedback, $E_{cl}(s) = R(s) - Y_{cl}(s) = \frac{1 + (H - 1)GD_{cl}}{1 + HGD_{cl}} R = (1 - \mathcal{T}(s))R(s)$

- $e_{ss} = \lim_{s\to 0} s(1 - \mathcal{T}(s))R(s) = \lim_{s\to 0} \frac{1 - \mathcal{T}(s)}{s^k}$
- We have to explicitly check the type by finding the largest value of k that keeps e_{ss} finite
- However, the relationship between the position/velocity/acceleration constants and the steady state error still holds

- Typing a system can also be done with respect to regulation, i.e. setting $R = V = 0$ and finding the highest order of disturbance W that leads to a finite steady state error; in this case the type is determined by the number of zeroes in the error transfer function
  - $E_{cl}(s) = R(s) - Y(s) = -\frac{G(s)}{1 + H(s)G(s)D_{cl}(s)}W \implies \frac{E_{cl}(s)}{W(s)} = -\frac{G(s)}{1 + H(s)G(s)D_{cl}(s)} = -T_w(s)$
    * Note the negative sign in the definition, so that $Y(s) = T_w(s)W(s)$
  - The type is the number of zeroes of $T_w(s)$ at the origin (instead of poles!)
  - Let $W(s) = \frac{1}{s^{k+1}}$ and $T_w(s) = s^m\tilde{T}_w(s)$ where $\tilde{T}_w(0) = \frac{1}{K_{m,w}}$
  - $-e_{ss} = y_s s = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sT_w(s)W(s) = \lim_{s\to 0} \tilde{T}_w(s)\frac{s^m}{s^k}$
  - Now we can see that $m > k \implies y_s s \to 0$, $m < k \implies y_{ss} \to \infty$ and $m = k \implies y_s s = \frac{1}{K_{m,w}}$
- Generally, the type of a system with respect to tracking can be different than the type with respect to regulation, so we must specify when stating the type
- We can also define a transfer function in terms of the noise, $\frac{Y(s)}{V(s)} = -H(s)\mathcal{T}(s) = T_v(s)$, assuming $R = W = 0$
  - For the noise however the use of a polynomial input is less realistic, since noise is usually very high in frequency