Lecture 13, Feb 27, 2024

Control System Performance

- In open-loop control, we control the plant without using feedback from its output
- For open-loop control, $Y_{ol} = GD_{ol}R + GW$
  - $E_{ol} = R - Y_{ol} = (1 - GD_{ol})R - GW$
  - Assuming no disturbance so $W(s) = 0$, we can define the open-loop transfer function
  - $T_{ol}(s) = \dfrac{Y(s)}{R(s)} = G(s)D_{ol}(s)$

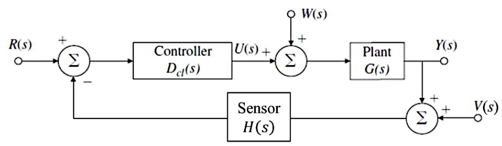

Figure 1: Closed-loop control.

- For closed-loop control, $Y_{cl} = \mathcal{T}R + G\mathcal{S}W - H\mathcal{T}V$
  - $E_{cl} = R - Y_{cl} = (1 - \mathcal{T})R - G\mathcal{S}W + H\mathcal{T}V$
    - Notice the negative sign in front of the W term, since the output is subtracted from reference
  - In this case, $V(s)$ is noise in our sensor measurements, which we separate from $W(s)$
    - $W(s)$ is usually low-frequency
    - $V(s)$ is usually high-frequency
  - Since we have an LTI system, we can consider the input and sources of noise separately
    - Assuming $W(s) = V(s) = 0$ we can define the closed-loop transfer function
      - $\mathcal{T}(s) = \dfrac{Y(s)}{R(s)} = \dfrac{G(s)D_{cl}(s)}{1 + H(s)G(s)D_{cl}(s)} = T_{cl}(s)$
    - Assuming $R(s) = V(s) = 0$ we can define the transfer function for process noise
      - $\dfrac{Y(s)}{W(s)} = G(s) \cdot \dfrac{1}{1 + H(s)G(s)D_{cl}(s)} = G(s)\mathcal{S}(s)$
      - Recall that $\mathcal{S}(s)$ is the *sensitivity* transfer function
    - Assuming $R(s) = W(s) = 0$ we can define the transfer function for measurement noise
      - $\dfrac{Y(s)}{V(s)} = -H(s) \cdot \dfrac{D_{cl}(s)G(s)}{1 + H(s)D_{cl}(s)G(s)} = -H(s)\mathcal{T}(s)$

Stability

- Consider an unstable plant $G(s) = \dfrac{b(s)}{a(s)}$, i.e. $a(s)$ has roots in the RHP; how can we design our controller to make the system stable?
  - Let the controller be $\dfrac{c(s)}{d(s)}$
  - For the open loop control we have $T_{ol} = \dfrac{b(s)}{a(s)}\dfrac{c(s)}{d(s)}$
    - Theoretically we can design $c(s)$ to cancel the unstable roots of $a(s)$, but as previously mentioned, this is impractical
    - We can make the same argument for cancelling bad zeroes (zeroes with small real part causing large overshoot)
      - For a nonminimum-phase zero, we can't do this at all because we'd need an unstable pole in the controller
    - Therefore in practice we can't use open-loop control to stabilize a plant
  - For closed-loop controllers, assume $H(s) = 1$ for the sensor, then $T_{cl} = \dfrac{b(s)c(s)}{a(s)d(s) + b(s)c(s)}$
    - Now we have a lot more options for eliminating the unwanted poles

- Example: Inverted pendulum (segway)
  - $\begin{cases}(m_t+m_p)\ddot{x}+b\dot{x}-m_pl\ddot{\theta}_0=u\\(I+m_pl^2)\ddot{\theta}-m_pgl\theta-m_pl\ddot{x}=0\end{cases}$
    * m_t, m_p are the masses of the cart and pendulum, I is the moment of inertia of the pendulum, l is the length of the pendulum, x is the cart's displacement and θ is the angle of the pendulum from normal
  - $G(s)=\dfrac{\Theta(s)}{U(s)}=\dfrac{m_pls}{((m_t+m_p)(I+m_pl^2)-m_p^2l^2)s^3+b(I+m_pl^2)s^2-m_pgl(m_t+m_p)s-m_pgbl}$
  - If we assume $b=0$ then we get a send order system with $((m_t+m_p)(I+m_pl^2)-m_p^2l^2)s^2-m_pgl(m_t+m_p)$ in the denominator
    * We can immediately tell that this is unstable by the RH criterion since since the s^1 term is missing
  - Assume $m_p=1\,\mathrm{kg}, I=1\,\mathrm{kgm}^2, l=1\,\mathrm{m}, m_t=0$ then we get $G(s)=\dfrac{1}{s^2-10}=\dfrac{1}{(s+3.16)(s-3.16)}$
  - Consider a controller $D_{cl}(s)=\dfrac{K(s+\gamma)}{s+\delta}$ and $H(s)=1$
    * Choosing $\gamma=-3.16$ cancels the RHP pole, but this is impractical
    * Choose $\gamma=+3.16$ cancels the stable pole, leaving $\dfrac{K}{(s-3.16)(s+\delta)+K}$
    * Now we can choose δ and K to move both poles of this second-order system to the LHP

Tracking

- We want to make the output follow the reference input as closely as possible, in effect having a unity transfer function from reference to output
- For open-loop control, we again have $T_{ol}=\dfrac{b(s)}{a(s)}\dfrac{c(s)}{d(s)}$
  - Designing the controller to cancel the plant's transfer function is only possible under the constraints:
    * The plant needs to be stable (and stable poles cannot be too close to the imaginary axis)
      - Trying to cancel out stable poles close to the imaginary axis may make the system too sensitive and cause unstable transients
    * The plant should have no zeroes in the RHP (since we'd need an RHP pole to cancel that)
    * The controller transfer function must be proper so it can be physically realized (it must be causal)
      - If the plant is strictly proper, this can be an issue since the controller would have to be improper
      - Digital controllers may be an exception
    * The controller cannot go beyond the plant's actuation limit (the response can't be too fast, or excite plant's resonance modes)
      - This will cause the system to be no longer linear
- For a closed-loop control system, most of the same restrictions apply, but we have more freedom to tune the response

Regulation

- Regulation is the ability of the control system to keep the error small when the input is constant, with added disturbances/noise
- In the open-loop case, the controller has no influence whatsoever on the effect of $W(s)$ on the output
- For the closed-loop controller: $E_{cl}=(1-\mathcal{T})R-G\mathcal{S}W+H\mathcal{T}V$
  - $E_{cl}=\dfrac{1+G(s)D_{cl}(s)(H(s)-1)}{1+H(s)G(s)D_{cl}(s)}R-\dfrac{G(s)}{1+H(s)G(s)D_{cl}(s)}W+\dfrac{H(s)G(s)D_{cl}(s)}{1+H(s)G(s)D_{cl}(s)}V$
  - Notice that if D is large, the second term is small so effect of W is small, but the third term gets closer to 1, so the effect of V is not reduced
  - Conversely if D is small we have less effect of V but more of W

    - To address this, we can design $D_{cl}(s)$ to have large values at low frequencies and small values at high frequencies, since W is often low frequency and V is often high

Sensitivity

- The robustness of the system against variations in the plant behaviour
- Assume that the plant transfer function can change from $G(s)$ to $G(s) + \delta G(s)$
- The *sensitivity* of the (overall) system transfer function T to plant G is defined as $\mathcal{S}_G^T = \frac{\frac{\delta T}{T}}{\frac{\delta G}{G}} = \frac{G}{T} \cdot \frac{\delta T}{\delta G}$
    - This is the ratio of the normalized change to the overall transfer function to the normalized change to the plant transfer function
- For open-loop control:
    - $T_{ol} + \delta T_{ol} = D_{ol}(G + \delta G) = D_{ol}G + D_{ol}\delta G = T_{ol} + D_{ol}\delta G = T_{ol} + \frac{T_{ol}}{G}\delta G \implies \delta T_{ol} = \frac{T_{ol}}{G}\delta G$
        * $\frac{\delta T_{ol}}{T_{ol}} = \frac{\delta G}{G}$ so the sensitivity is 1
    - i.e. whatever change happens in the plant, it will be immediately reflected in the entire system
- For closed-loop control, $\mathcal{S}_G^{T_{cl}} = \frac{G}{T_{cl}}\frac{\delta T_{cl}}{\delta G} = \frac{G}{T_{cl}} \cdot \frac{\mathrm{d}T_{cl}}{\mathrm{d}G}$
    - We can show that $\mathcal{S}_G^{T_{cl}} = \frac{1}{1 + HGD_{cl}}$
        * This is why we define the sensitivity transfer function as $\mathcal{S} = \frac{1}{1 + HGD_{cl}}$
        * The sensitivity is not 1 but is mitigated by the additional term in the denominator
        * The larger the controller D_{cl}, the more robust it is to changes in the plant
    - The complementary sensitivity transfer function is $\mathcal{T} = \frac{GD_{cl}}{1 + HGD_{cl}}$
        * Notice that this is just the closed-loop transfer function
        * This is named so because for the case of a perfect sensor $H(s) = 1$, $\mathcal{S} + \mathcal{T} = 1$