Lecture 12, Feb 15, 2024

Stability of LTI Systems

- For a transfer function $H(s) = K_H \dfrac{\prod_{i=1}^{m}(s - z_i)}{\prod_{j=1}^{n}(s - p_j)}$, the impulse response will look like a sum of exponentials, $y(t) = \sum_{j=1}^{n} C_j e^{p_j t}$ (assuming poles are distinct)
  - The coefficients C_j depend on the initial conditions and locations of zeroes
  - If a pole is repeated k times, we will have terms $C_{j_0} e^{p_j} + C_{j_1} t e^{p_j} + \cdots + C_{j_{k-1}} t^{k-1} e^{p_j}$
  - The system response is bounded if and only if all $\mathrm{Re}(p_j) \leq 0$ (regardless of repeating poles); hence any poles in the RHP are unstable

Definition

We define three types of stability for systems:
- *Bounded-Input-Bounded-Output* (BIBO) Stability: Any bounded input generates a bounded output (with no requirement on convergence).
- *Asymptotic* Stability: Any inital condition generates an output which approaches for zero as time approaches infinity.
- *Marginal* (or *Neutral*) Stability: Any initial condition generates an output which is bounded (for a zero input).

- Asymptotic stability is a generally stronger form of stability than BIBO
  - All asymptotically stable systems are also BIBO stable
- For all LTI systems, all BIBO systems are also asymptotically stable
- If any poles are exactly on the imaginary axis, then if they are non-repeating, the system is marginally/neurally stable, but if they are repeating, then the system is unstable
  - This will result in either a constant output or a free oscillator
- The *Routh-Hurwitz* stability criterion can be used to identify the stability of a system without explicitly factoring the characteristic equation
  - Consider the characteristic equation $s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$
  - If all poles are in the LHP, then all coefficients a_i are positive and real
    * Therefore if any $a_i \leq 0$, then the system is always unstable (or marginally stable)
  - If all $a_i > 0$, we need to form the *Routh array* to check for stability
    * The array consists of $n + 1$ rows, with row i corresponding to s^i
    * Row n contains $1, a_2, a_4, \ldots$
    * Row $n - 1$ contains $a_1, a_3, a_5, \ldots$
    * Row $n - 2$ contains $b_1, b_2, b_3, \ldots$ where:
      - $b_1 = -\dfrac{1}{a_1} \det \begin{bmatrix} 1 & a_2 \\ a_1 & a_3 \end{bmatrix}$
      - $b_2 = -\dfrac{1}{a_1} \det \begin{bmatrix} 1 & a_4 \\ a_1 & a_5 \end{bmatrix}$
      - $b_3 = -\dfrac{1}{a_1} \det \begin{bmatrix} 1 & a_6 \\ a_1 & a_7 \end{bmatrix}$
    * Row $n - 3$ contains $c_1, c_2, c_3, \ldots$ where:
      - $c_1 = -\dfrac{1}{b_1} \det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_2 \end{bmatrix}$
      - $c_2 = -\dfrac{1}{b_1} \det \begin{bmatrix} a_1 & a_5 \\ b_1 & b_3 \end{bmatrix}$
      - $c_2 = -\dfrac{1}{b_1} \det \begin{bmatrix} a_1 & a_7 \\ b_1 & b_4 \end{bmatrix}$
    * For x_i, consider the 2×2 matrix formed by taking column 1 and column $i + 1$ of the two previous rows, take the negative of its determinant and divide by the bottom left entry

        - We treat any missing entries as zeroes
        - This means for each row starting from $n-2$, we will get one fewer element (zero entry) per row
        - By row 1 we are left with only one element
        - For row 0 we still have one element, obtained by treating the missing entry as a zero when calculating the determinant
        - Past row 0 all entries will be zero, so we stop
    * Note: all elements of each row can be divided by a common factor to simplify computation
  - A system is stable if and only if all elements in the first column of the Routh array are positive
  - The number of roots in the RHP is equal to the number of sign changes in the first column of the Routh array
  - Note the two special cases:
    * One of the elements in the first column is zero
        - Replace this element by some small positive value $\text{var}\,\epsilon$ and construct the rest of the array
        - Take the limit $\text{var}\,\epsilon \to 0^+$ and check the sign of the first column
    * An entire row is zero
        - Take the contents of the row above this, and create an auxiliary polynomial with only even powers, using the row as coefficients
        - e.g. if the row above the zero row has 3 and 12, then the auxiliary polynomial is $p(s) = 3s^2 + 12$
        - Differentiate this polynomial and use the coefficients in the derivative as the new contents for the zero row
- Example: unity feedback system, with a PI controller $K + \dfrac{K_I}{s}$, and a plant $\dfrac{1}{(s+1)(s+2)}$
  - For what values of K and K_I is the closed-loop system stable?
  - $H(s) = \dfrac{\left(K + \frac{K_I}{s}\right)\left(\frac{1}{(s+1)(s+2)}\right)}{1 + \left(K + \frac{K_I}{s}\right)\left(\frac{1}{(s+1)(s+2)}\right)} = \dfrac{Ks + K_I}{s^3 + 3s^2 + (2+K)s + K_I}$
  - From this we can see that a necessary condition is $K_I > 0$ and $K > -2$, but this is not a sufficient condition
  - Form the Routh array:
    *

Row 3	1	$(2+K)$
Row 2	3	K_I
Row 1	$(6+3K-K_I)/3$	
Row 0	K_I	

  - To have all terms in the first row be positive, we require $K_I > 0$ and $K > \dfrac{1}{3}K_I - 2$
  - Note that for this system, even if we took $K_I = 0$, because we'd have a pole at 0 and zero at 0, they cancel out and the overall system is still stable