Lecture 10, Feb 8, 2024

Second Order System Response (Continued)

- Consider the step response
  - $y_s(t) = \mathcal{L}^{-1}\left\{\frac{H(s)}{s}\right\}$
    $= \mathcal{L}^{-1}\left\{\frac{\sigma^2+\omega_d^2}{((s+\sigma)^2+\omega_d^2)s}\right\}$
    $= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+2\sigma}{(s+\sigma)^2+\omega_d^2}\right\}$
    $= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+\sigma}{(s+\sigma)^2+\omega_d^2} - \frac{\sigma}{\omega_d}\frac{\omega_d}{(s+\sigma)^2+\omega_d^2}\right\}$
    $= 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$
    $= 1 - e^{-\sigma t}\frac{\omega_n}{\omega_d}\cos(\omega_d t - \theta)$
    - Where $\theta = \tan^{-1}\left(\frac{\omega_d}{\sigma}\right) = \tan^{-1}\left(\frac{\omega_d}{\zeta\omega_n}\right)$
- For an overdamped system, the two separate poles lie on the real axis, and with decreasing ζ the poles move together until they overlap, and then move radially into the imaginary axis
- The system starts with no oscillation but a slow response to faster responses but oscillations begin; when $\zeta = 0$ the poles are purely imaginary, at which point the response is purely oscillatory and no decay occurs
  - When $\zeta = 1$, the poles overlap, and we get critical damping, which is the fastest possible system response without oscillation

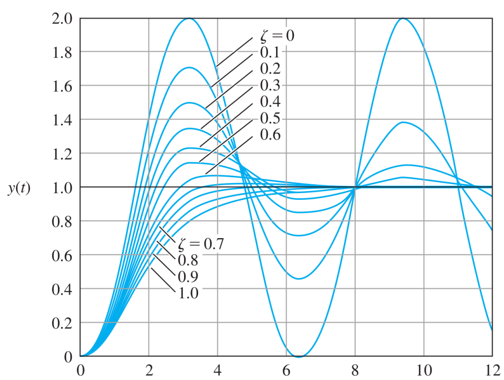

Figure 1: Step response of an underdamped second-order system.

- For the second order system, we can characterize it using the following (for a unit step input):
  - DC gain y_{ss}: the steady-state value of the system output
  - Peak time t_p: time to reach the maximum overshoot/undershoot point
  - Overshoot M_p: the max amount the output overshoots $y_s s$, divided by the steady state value (usually as a percentage)
  - Rise time t_r: the time the system takes to rise from 10% to 90% of $y_s s$
  - Settling time t_s: the time the system takes to reach, and stay within, 1% of $y_s s$ (2% in some texts)
- DC gain: $y_{ss} = \lim_{s\to 0} sY_s(s) = \lim_{s\to 0} s\frac{1}{s}H(s) = \frac{\omega_n^2}{\omega_n^2} = 1$
  - The DC gain here is 1 because when we derived the system, we multiplied u by k
  - Without this scaling the DC gain would be k instead

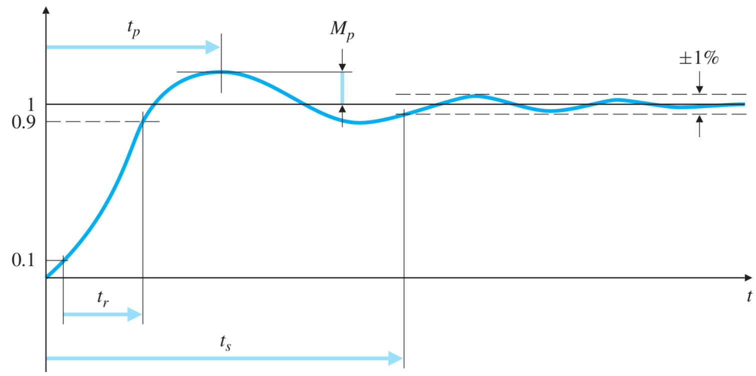

Figure 2: Illustration of the characteristics of second-order system response.

- Peak time:
  - Take derivative: $\dot{y}_s(t) = \mathcal{L}^{-1}\{sY_s(s)\} = \mathcal{L}^{-1}\left\{s\frac{1}{s}H(s)\right\} = y_i(t)$
    - Note the derivative of the step response is just the impulse response
  - Therefore $y_i(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2 t}\right)$
  - $\dot{y}_s(t) = y_i(t) = 0 \implies \omega_n\sqrt{1-\zeta^2 t} = n\pi \implies t = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}}$
  - The first peak occurs at $t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$
  - As we reduce the damping,
- Overshoot:
  - Substitute t_p into the step response to get the peak of the response
  - $y_s(t_p) = 1 - \frac{e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\cos(\pi - \theta) = 1 + \frac{e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\cos\theta$
  - We know $\cos\theta = \sqrt{1-\zeta^2}$ so this simplifies to $1 + e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$
  - The overshoot is therefore $M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$ (or times 100 for percentage)
  - Notice that this depends only on ζ
    - Usually we're interested in two values: $\zeta = \frac{1}{2}$ which gives 16% overshoot, and $\zeta = 0.7$ which gives 5% overshoot
  - The percent overshoot decreases with ζ, but $\omega_n t_p$ increases with ζ
- Rule of thumb: the response of the second-order underdamped systems (with no finite zeroes) with different damping ratios rises roughly with the same pace
  - Typically we related t_r to only ω_n instead of also ζ, as an approximation
  - For $\zeta = 0.5$, we can approximate $t_r \approx \frac{1.8}{\omega_n}$
  - We typically choose ζ between 0.5 and 0.7 for a balance between overshoot and rise time
- For settling time we can approximate the deviation of the response by the exponential only
  - Therefore $e^{-\zeta\omega_n t_s} \approx 0.01 \implies t_s \approx \frac{4.6}{\zeta\omega_n} = \frac{4.6}{\sigma}$

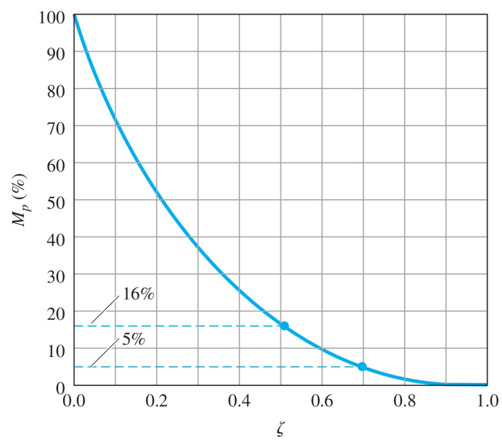

Figure 3: Overshoot as a function of damping ratio.