Lecture 5, Jan 18, 2023

Independence

Definition

Two events A and B are *independent* if $P(A|B) = P(A)$ or $P(B|A) = P(B)$

- Note, $P(B|A) = P(B) = \dfrac{P(A \cap B)}{P(A)} \implies P(A \cap B) = P(A)P(B) \implies P(A|B) = \dfrac{P(A)P(B)}{P(B)} = P(A)$
- i.e. Two events are independent if one of them happening does not affect the probability of the other
- Note: Independence is not the same as mutual exclusion! e.g. for a coin flip, heads and tails are mutually exclusive, but they are not independent since $P(H|T) = 0 \neq P(H)$

Bayes' Rule

- Suppose $P(A) > 0, P(B) > 0$; we know $P(A \cap B) = P(A|B)P(B)$ and $P(A \cap B) = P(B|A)P(A)$ so $P(A|B)P(B) = P(B|A)P(A)$
- Rearrange to get $\dfrac{P(A|B)}{P(A)} = \dfrac{P(B|A)}{P(B)}$

Definition

Bayes' Rule:

$$\frac{P(A|B)}{P(A)} = \frac{P(B|A)}{P(B)}$$

or

$$P(A|B)P(B) = P(B|A)P(A)$$

- Bayes' rule is useful for reversing causality; based on what we saw, we can make inferences about what caused it

Partitions and Probability

- Recall $B_1, \cdots, B_k$ is a partition if $B_i \cap B_j = \varnothing$ whenever $i \neq j$ and $B_1 \cup B_2 \cdots \cup B_k = S$

Equation

The law of total probability: Given a partition $B_1, \cdots, B_k$ and some event A,

$$\begin{aligned} P(A) &= \sum_{i=1}^{k} P(A \cap B_i) \\ &= \sum_{i=1}^{k} P(A|B_i)P(B_i) \end{aligned}$$

- Proof:
  - Observe $A = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_k)$
  - Then $P(A) = P((A \cap B_1) \cup \cdots \cup (A \cap B_k))$
  - Note when we partitioned the sample space, each B_i is mutually exclusive, therefore $A \cap B_i$ are also mutually exclusive
  - Therefore $P(A) = P(A \cap B_1) + \cdots + P(A \cap B_k)$ by mutual exclusivity

  - $P(A) = \sum_{i=1}^{k} P(A \cap B_i)$
- Example: Machines 1, 2, 3 make products 30%, 45%, and 25% of the time respectively (as a partition); the product is defective 2%, 3%, and 2% of the time for these machines respectively, what is the probability that the product is defective
  - $P(\text{defective}) = P(\text{defective}|\text{made by 1})P(\text{made by 1}) + P(\text{defective}|\text{made by 2})P(\text{made by 2}) + P(\text{defective}|\text{made by 3})P(\text{made by 3})$
  - $P(\text{defective}) = 0.30 \cdot 0.02 + 0.45 \cdot 0.03 + 0.25 \cdot 0.02 = 2.45\%$
- In general independence, partitions, and conditional probability show up in the problem statement and it's up to us to interpret them