Lecture 36, Apr 12, 2023

Markov Chains

- Let us have n states, and $p_i(k)$ which gives the probability of being in state i in time k, for $i = 1, \dots, n$; at any time if we're in state i, then there is a probability P_{ij} of moving to state j
  - Therefore $\sum_{i=1}^{n} p_i(k) = 1$ since the system must be in some state at any time
  - $\sum_{j=1}^{n} P_{ij} = 1$ since the system must go somewhere in the next time (this includes itself, i.e. $P_{ii} \geq 0$)
- Observe that $p_i(k+1) = \sum_{j=1}^{n} p_j(k) P_{ji}$, i.e. the probability of being in i at the next time is the sum of the probabilities of all states transitioning to i
- Let $\boldsymbol{p}(k) = \begin{bmatrix} p_1(k) \\ \vdots \\ p_n(k) \end{bmatrix}, \boldsymbol{M} = \begin{bmatrix} P_{00} & \cdots & P_{n0} \\ \vdots & \ddots & \vdots \\ P_{0n} & \cdots & P_{n0} \end{bmatrix}$, then $\boldsymbol{p}(k+1) = \boldsymbol{M}\boldsymbol{p}(k)$, analogous to an LTI system
  - Generally $\boldsymbol{p}(k+s) = \boldsymbol{M}^s \boldsymbol{p}(k)$
  - Notice that $\mathbf{1}^T \boldsymbol{M} = \mathbf{1}^T$ where $\mathbf{1}$ is the vector of all 1s, so it's a left eigenvector with eigenvalue 1
- Any eigenvector $\boldsymbol{q} = \boldsymbol{M}\boldsymbol{q}$ is a steady state, which the system will never come out of once entered
  - We can show that $\boldsymbol{q} = \lim_{k \to \infty} \boldsymbol{M}^k \boldsymbol{p}$ for any initial PMF $\boldsymbol{p}$, if such $\boldsymbol{q}$ exists
  - Intuitively if we just let the Markov chain do its thing eventually it'll end up in a steady state
- Example: Suppose we have 2 states, on (1) or off (0), $P_{11} = 0.99, P_{10} = 0.01, P_{01} = 0, P_{00} = 1$
  - $\boldsymbol{M} = \begin{bmatrix} 1 & 0.01 \\ 0 & 0.99 \end{bmatrix}$
  - Steady state is $\boldsymbol{p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, i.e. 100% chance of off