Lecture 33, Apr 3, 2023

One vs. Two-Tailed Tests

- *Two-tailed tests* are for hypotheses for an exact value (or two-sided range) of a variable; *one-tailed tests* are for hypotheses with one-sided ranges
- So far we've only considered two-tailed tests (e.g. temperature tomorrow is 10°C; H_0 is temperature is 10 degrees, H_1 is temperature is not)
- A one-tailed test would be e.g. hypothesis is eating something would cause you to live past 80, then $H_0 : \theta \leq 80, H_1 : \theta > 80$, we can set critical region $\theta > 82$

Relation of Hypothesis Testing to Confidence Intervals

- Consider a scenario where $H_0 : \mu = \mu_0, H_1 : \mu \neq \mu_0$, for a sample of size n with known variance σ^2
- Let $Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$ as usual
- Before, we set the critical region and then calculated the probability of a type I error; what happens if we set α first and then calculate the critical region?
- $\alpha = P(\text{Type I}) = P(Z \leq -z_{\alpha/2} \cup Z \geq z_{\alpha/2}) \implies 1 - \alpha = P\left(-z_{\alpha/2} \leq \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \leq z_{\alpha/2}\right)$
- The critical region would be $\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \notin [-z_{\alpha/2}, z_{\alpha/2}]$
- This is the exact same as a confidence interval

p-Values

Definition

Given a sample $X_1, \cdots, X_n$ with variance σ^2, and the null hypothesis $H_0 : \mu = \mu_0$, then the p-value is defined as

$$\begin{aligned} p &= P\left(\left|\frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}}\right| > |z|\right) \\ &= P(|Z| \geq |z|) \\ &= 2P(Z > |z|) \\ &= 2(1 - \Phi(|z|)) \end{aligned}$$

where $Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$ has distribution $n(z; 0, 1)$ and z is the observed value of Z

- The p value is the probability that, given H_0 is true, an even more extreme observation will be made
  - If $p \approx 0$, that means z is very far from 0 and therefore $\bar{x}$ is very far from μ_0, so H_0 is highly unlikely
  - If $p \approx 1$, then z is close to 0 so $\bar{x} \approx \mu$, so H_0 cannot be rejected
- We still need to assume H_0, but unlike the other methods of hypothesis testing, to compute a p-value we don't need to make an ad-hoc critical region
- Example: $H_0 : \mu = 5, H_1 : \mu \neq 5$, with $n = 40, \bar{x} = 5.5, s = \sigma = 1$
  - $z = \frac{5.5 - 5}{\frac{1}{\sqrt{40}}} \approx 3.16$
    - If we choose $\alpha = 0.05$ so the critical region for Z is outside $[-1.96, 1.96]$, then z is outside the critical region so we reject H_0
  - $p = 2P(Z > |z|) = 2(1 - \Phi(3.16)) = 0.0016 \approx 0$
    - The very low p-value means H_0 is highly unlikely

– Note to compute the p-value we didn't need to specify a critical region