Lecture 3, Jan 13, 2023

Partitions

Definition

A *partition* is a way to divide a number of elements into groups of certain sizes, where we don't care about the order of elements in the groups

The number of partitions of n distinct objects with m bins of size $n_1, \cdots, n_m$ is

$$\binom{n}{n_1, n_2, \cdots, n_m} = \frac{n!}{n_1!n_2!\cdots n_m!}$$

- Example: Partitions of $\{a, b, c\}$ with $n_1 = 2, n_2 = 1$
  - Possible partitions are $\{ab, c\}, \{ac, b\}$ or $\{bc, a\}$, so there are 3 partitions
  - Notice $3 = \dfrac{3!}{2!1!}$
- Formally, let there be n distinct objects and m bins which can hold $n_1, n_2, \cdots, n_m$ objects such that $\sum_{k=1}^{m} n_k = n$, then the number of partitions is $\binom{n}{n_1, n_2, \cdots, n_m} = \dfrac{n!}{n_1!n_2!\cdots n_m!}$
  - This is the same calculation as permutations with identical objects
  - If $n_k = 1$, then the number of partitions is $n!$ as we just have a permutation; then we divide through by the number of possible permutations within each box since we don't care about the order of elements within boxes

Combinations

- A combination is a permutation where we don't care about the order
- Given n objects, the number of subsets of size r we can make is $\binom{n}{r} = \binom{r}{r, n-r} = \dfrac{n!}{(n-r)!r!}$
  - This is known as "n choose r"
- In terms of partitions, we have one box of size r (the elements we're choosing), and another box of size $n - r$ (the elements we're not choosing)

Probability

- Given a sample space S and an event $A \subseteq S$, then the probability of event A is $P(A) \in [0, 1]$
  - $P(S) = 1$
  - If $A \cap B = \varnothing$, then $P(A \cup B) = P(A) + P(B)$
- Example: What's the probability of getting 2 aces and 3 jacks in a hand?
  - Ways to get 2 aces: $\binom{4}{2} = 6$
    - Out of 4 aces in the deck we're picking 2 of them
  - Ways to get 3 jacks: $\binom{4}{3} = 4$
  - Therefore we have 24 such hands, so the probability is $\dfrac{24}{\binom{52}{5}} = 0.9 \times 10^{-5}$