Lecture 17 (Online)

Moments and Moment-Generating Functions

Definition

The rth *moment* about the origin of the random variable X is

$$\mu'_r = E[X^r] = \begin{cases} \displaystyle\sum_x x^r f(x) & X \text{ discrete} \\ \displaystyle\int_{-\infty}^{\infty} x^r f(x)\,\mathrm{d}x & X \text{ continuous} \end{cases}$$

- The first moment is the mean: $\mu = \mu'_1$
- The second moment is related to variance: $\sigma^2 = E[X^2] - \mu^2 = \mu'_2 - \mu^2$

Definition

The *moment-generating function* of the random variable X is

$$M_X(t) = E[e^{tX}] = \begin{cases} \displaystyle\sum_x e^{tx} f(x) & X \text{ discrete} \\ \displaystyle\int_{-\infty}^{\infty} e^{tx} f(x)\,\mathrm{d}x & X \text{ continuous} \end{cases}$$

- Consider the discrete case: $\displaystyle \left.\frac{\mathrm{d}^r M_X(t)}{\mathrm{d}t^r}\right|_{t=0} = \left.\frac{\mathrm{d}^r}{\mathrm{d}t^r} \sum_x e^{tx} f(x)\right|_{t=0}$

$$\begin{aligned} &= \sum_x f(x) \left.\frac{\mathrm{d}^r}{\mathrm{d}t^r}\right|_{t=0} \\ &= \sum_x f(x) x^r \left. e^{tx}\right|_{t=0} \\ &= \sum_x f(x) x^r \\ &= E[X^r] \\ &= \mu'_r \end{aligned}$$

  - This works the same in the continuous case
- In general $\displaystyle \mu'_r = \left.\frac{\mathrm{d}^r M_X(t)}{\mathrm{d}t^r}\right|_{t=0}$

Linear Combinations of Random Variables

- Consider a discrete RV X with distribution $f(x)$; let $Y = aX$, then the distribution $h(y) = f\left(\frac{y}{a}\right)$, using the formula we found before
- In the continuous case using the formula before $h(y) = \frac{1}{|a|} f\left(\frac{y}{a}\right)$
- If we have the moment generating function of X as $M_X(t)$, how do we find $M_Y(t)$?

- $M_Y(t) = \int_{-\infty}^{\infty} e^{ty} h(y)\,\mathrm{d}y = \frac{1}{|a|}\int_{-\infty}^{\infty} e^{ty} f\left(\frac{y}{a}\right)\,\mathrm{d}y$
  $= \frac{1}{|\alpha|}\int_{-\infty}^{\infty} e^{taz} f(z) a\,\mathrm{d}z$
  $= \int_{-\infty}^{\infty} e^{taz} f(z)\,\mathrm{d}z$
  $= M_X(at)$
  - This is also true in the discrete case
- In general $M_{aX} = M_X(at)$
- What about a sum of independent RVs $Z = X + Y$?
  - $h(z) = P(X + Y = z) = \sum_w P(X = w)P(Y = z - w) = \sum_{w=-\infty}^{\infty} f(w)g(z - w)$
  - In the continuous case this is similar: $h(z) = \int_{-\infty}^{\infty} f(w)g(z - w)\,\mathrm{d}w = (f * g)(z)$
    * This is a convolution
  - $M_Z(t) = \sum_z e^{tz} h(z) = \sum_z e^{tz} \sum_w f(w)g(z - w) = \sum_w f(w) \sum_z e^{tz} g(z - w)$
    * Let $k = z - w$
    * $M_Z(t) = \sum_w f(w) \sum_k e^{t(k+w)} g(k) = \sum_w e^{tw} f(w) \sum_k e^{tk} g(k) = M_X(t) + M_Y(t)$
- In general $M_{X+Y} = M_X(t)M_Y(t)$
- There is a connection between moment generating functions and Laplace/Fourier transforms