Lecture 13, Feb 8, 2022

Poisson Distribution

- Given a sequence of independent intervals, how many times will a given event occur within an interval?
- Example: number of goals in a soccer game, number of snow days in a year, number of arrivals per minute at an internet router
- A *Poisson process* is a process where the number of arrivals in an interval is a random variable independent of the other intervals
  - The number of arrivals is proportional to the length of the interval
  - The number of arrivals in a given interval follows a Poisson PMF

Definition

The *Poisson PMF* is given by

$$p(x; \lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, \cdots$$

where λ is the Poisson parameter and x is the number of arrivals in the interval described by the distribution

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The Possion parameter λ is both the mean and the variance of the Possion distribution; we have

$$\lambda = rt$$

where r is the rate of arrivals, and t is the length of the interval

- Statistics of the Poisson PMF:
  - $\mu = \sum_{x=0}^{\infty} x\frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^x}{(x-1)!} = e^{-\lambda}\lambda\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!} = e^{-\lambda}\lambda e^{\lambda} = \lambda$
  - We can also show $\sigma^2 = \lambda$ through a similar calculation
- We can think of the Poisson distribution as the binomial distribution, in the limit where $n \to \infty, p \to 0$ and $np \to \lambda$
  - $$\begin{aligned}\lim_{n\to\infty,p\to 0}\binom{n}{x}p^x(1-p)^{n-x} &= \lim_{n\to\infty,p\to 0}\frac{n(n-1)\cdots(n-x+1)}{x!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x} \\ &= \lim_{n\to\infty,p\to 0}\frac{n^x}{x!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x} \\ &= \frac{\lambda^x}{x!}\lim_{n\to\infty,p\to 0}\left(1-\frac{\lambda}{n}\right)^{n-x} \\ &= \lim_{n\to\infty,p\to 0}\left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-x} \\ &= \lim_{n\to\infty,p\to 0} e^{-\lambda}\cdot 1 \\ &= \frac{e^{-\lambda}\lambda^x}{x!} \\ &= p(x;\lambda)\end{aligned}$$