Lecture 11 (Recorded)

Expectations of Linear Combinations of Random Variables

Definition

A function $p(x)$ is linear if $p(ax + y) = ap(x) + p(y)$

- Because integration and summation are linear, expectation $E[X]$ is linear: $E[aX + Y] = aE[X] + E[Y]$
- Useful implications:
  - $E[aX + b] = aE[X] + b$
  - $E[g(X, Y) + h(X, Y)] = E[g(X, Y)] + E[h(X, Y)]$

Variance and Independence

- If X and Y are independent:
$$\begin{aligned} E[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyg(x)h(y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty} xg(x)\,\mathrm{d}x \int_{-\infty}^{\infty} yh(y)\,\mathrm{d}y \\ &= E[X]E[Y] \end{aligned}$$
  - Therefore because $\sigma_{XY} = E[XY] - E[X]E[Y]$, this means independence implies uncorrelated
  - But uncorrelated does not imply independence!
- Note $\sigma^2_{aX+bY+c} = a^2\sigma^2_X + b^2\sigma^2_Y + 2ab\sigma^2_{XY}$
  - The constant does not affect the variance
  - There is a cross term
  - The cross term σ_{XY} disappears if X and Y are independent variables

Uniform Distribution

- Every element in the sample space has the same probability
- For $S = 1, \cdots, n$, then $f(k) = \frac{1}{n}$ for $k \in S$

Binomial Distribution

- A Bernoulli random variable has 2 outcomes $S = 0, 1$, and each has a probability (e.g. a coin flip)

Definition

A *Bernoulli process* is a process involving n repeated, independent, identical trials where the only outcomes possible are 1 and 0, with $P(1) = p$; let X be the number of 1's that occur, then the *Binomial distribution* is the probability mass function for X:

$$P(X = x) = f(x) = b(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x}$$

- Example: when sending a string of 0s and 1s, how many errors will occur?
- $p^x(1-p)^{n-x}$ is the probability of having x 1's and $n - x$ 0's in some order; $\binom{n}{x}$ is all the ways to have that many 1's and 0's
- $E[X] = np$ is the expectation value of a binomial distribution

- We can think of the Bernoulli process as a sum of n trials, $X = Y_1 + \cdots + Y_n$, so since expectation is linear, we can just add up the probabilities

- Similarly $\sigma_X^2 = np(1-p)$

Multinomial Distribution

- An extension of the binomial distribution where each trial can have m outcomes instead of just 2
- The chance of each outcome E_i is p_i
- x_i is the number of times we get outcome E_i; $\sum_i x_i = n$

Definition

The *multinomial distribution* is

$$f(x_1, \cdots, x_n; p_1, \cdots, p_m, n) = \binom{n}{x_1, x_2, \cdots, x_m} p_1^{x_1} p_2^{x_2} \cdots p_m^{x_m}$$