Lecture 32, Apr 3, 2023

Time-Varying Fields: Overview

- So far we've discussed only static charges and steady currents; for these cases, electricity and magnetism are separate entities
- With time-varying charges and currents, electricity and magnetism are now related by Maxwell's equations:
  - $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$
  - $\vec{\nabla} \times \vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t}$
  - $\vec{\nabla} \cdot \vec{D} = \rho_v$
  - $\vec{\nabla} \cdot \vec{B} = 0$
- Now changes in the electric field induce changes in the magnetic field and vice versa, which allows electromagnetic waves

Faraday's and Lenz's Laws

- A changing magnetic flux causes a current to flow in a closed loop; this means an electromotive force (EMF) is created
- Faraday's law states that the EMF induced in a circuit is directly proportional to the time rate of change of the magnetic flux linking that circuit
- The EMF is the amount of work done per unit charge, or $V_{emf} = \oint_C \dfrac{\vec{F}_e \cdot \mathrm{d}\vec{l}}{q} = \oint_C \vec{E} \cdot \mathrm{d}\vec{l}$
  - Because $\vec{F} = q\vec{E} + q\vec{u} \times \vec{B}$, an EMF can be caused by an electric force, a magnetic force, or a combination of both
  - Notice since the electric field is conservative in electrostatics, the EMF is zero without time-varying fields

Equation

Faraday's Law:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \iff V_{emf} = \oint_C \vec{E} \cdot \mathrm{d}\vec{l} = -\frac{\partial}{\partial t} \iint_S \vec{B} \cdot \mathrm{d}\vec{s} = -\frac{\partial \Phi}{\partial t}$$

- Electric fields induced by magnetic flux changes are not conservative, which is how EMF can be zero
- If the loop is closed, then the EMF will cause a current to flow, $I_{emf} = \dfrac{V_{emf}}{R}$
  - Lenz's law: The direction of this current is such that the magnetic field it produces opposes the original change in the magnetic field
  - Lenz's law is why there is a negative sign on Faraday's law
  - The field is not opposing the field, but opposing the change (e.g. if the field is up but decreasing, the induced current produces a field that still points up, to compensate the decrease)
- Lenz's law is a statement of the conservation of energy; if the induced current flowed the other way, it would lead to a positive feedback loop and violate conservation of energy
- With multiple turns in the loop, $V_{emf} = -N\dfrac{\partial \Phi}{\partial t} = -N\dfrac{\partial}{\partial t}\iint_S \vec{B} \cdot \mathrm{d}\vec{s}$ since every turn experiences the same EMF
- To cause this change in the flux, we could either change $\vec{B}$ itself or change the surface S (e.g. expand/shrink, change in orientation)
  - Changes in flux caused by changes in $\vec{B}$ causes an EMF known as the *transformer EMF*
    - This is created by the induced electric force
  - Changes in the flux caused by changes in S is known as a *motional EMF*

        * This is caused by moving charges in the presence of a $\vec{B}$

- Example: transformer EMF: find induced V_{emf} in a torus, if $I(t) = I_0 \cos(\omega t)$ passes through the centre, N turns of wire, inner diameter a, outer diameter b, height c
  - $\vec{B}(t) = \dfrac{\mu I(t)}{2\pi r}$
  - $\Phi(t) = \iint_S \vec{B}(t) \cdot \mathrm{d}\vec{s} = \int_a^b \int_{-c}^0 \dfrac{\mu I(t)}{2\pi r} \, \mathrm{d}z \, \mathrm{d}r = \dfrac{\mu c I(t)}{2\pi} \ln \dfrac{b}{a} = \dfrac{\mu c \ln\left(\frac{b}{a}\right)}{2\pi} I_0 \cos(\omega t)$
  - $V_{emf} = -N \dfrac{\partial \Phi}{\partial t} = \dfrac{N \mu c \ln\left(\frac{b}{a}\right) I_0 \omega}{2\pi} \sin(\omega t)$
  - Notice: $V_{emf} = V_0 \sin(\omega t) = L \dfrac{\mathrm{d}I}{\mathrm{d}t}$