Lecture 31, Mar 31, 2023

Mutual Inductance Example

- Mutual inductance is usually denoted $L_{12} = M = L_{21}$; the fact that the mutual inductances go both ways is a result of the *reciprocity* rule
- Example: small circular loop of radius a, a distance d from an infinite wire carrying I_1; what is the mutual inductance of 1 due to 2, L_{21}?
  - We can instead find L_{12} since we don't know how to find the flux through an infinitely long wire
  - Approximate flux from the infinite wire as $B_1 \approx \dfrac{\mu_0 I_1}{2\pi d}$ (i.e. take the field at the centre of the circle), because the loop is small
  - $\Phi_{12} \approx \dfrac{\mu_0 I_1}{2\pi d}(\pi a^2)$
  - $L_{12} = L_{21} = M = \dfrac{N_2 \Phi_{12}}{I_1} = \dfrac{\mu_0 a^2}{2d}$

Magnetic Energy

- Like how electric potential energy is the energy it took to build up a collection of charges, magnetic potential energy is the energy it took to create a current distribution
- For free current distributions, this is due to Lenz's law – the field will oppose a change in current
- For bound current distributions, this is the energy required to align the magnetic dipoles within the material

Definition

The stored *magnetic potential energy* of a current distribution is

$$W_m = \frac{1}{2}\iiint_v \vec{B} \cdot \vec{H} \,\mathrm{d}v = \frac{1}{2}\iiint_v \mu_r \mu_0 |\vec{H}|^2 \,\mathrm{d}v$$

The *magnetic energy density* is then

$$w_m = \frac{1}{2}\vec{B} \cdot \vec{H}$$

- Example: stored magnetic energy within an infinitely long solenoid
  - $H = nI, B = \mu_0 \mu_r n I$ by Ampere's law
  - $W_m = \dfrac{1}{2}\displaystyle\iiint_v \mu_r \mu_0 |\vec{H}|^2 \,\mathrm{d}v = \dfrac{1}{2}\mu_r \mu_0 (\pi a^2 l)(n^2 I^2) = \dfrac{\mu_r \mu_0 \pi a^2 N^2 I^2}{2l}$
- The energy stored in an inductive element is $W_m = \dfrac{1}{2}LI^2$, which holds here as well
  - Often inductance is easier to find by first finding the energy and then solving for L
- Example: energy storage in coupled circular toroids – what is their self inductance, mutual inductance, and the energy stored?
  - First find B_1, B_2 from Ampere's law, with a loop concentric to the toroids going through them
  - We will approximate $B_1 = \dfrac{N_1 I_1 \mu_r \mu_0}{2\pi r_0}$ by assuming a constant B through the cross section (so the expression isn't a nightmare)
  - $B_1 = \dfrac{N_2 I_2 \mu_r \mu_0}{2\pi r_0}$
  - We can write the energy as $W_m = \dfrac{1}{2}L_{11}I_1^2 + \dfrac{1}{2}L_{22}I_2^2 + \dfrac{1}{2}L_{12}I_1 I_2 + \dfrac{1}{2}L_{21}I_1 I_2$
    - The first 2 terms are the self-energies; the second 2 terms are the mutual energies
  - To find L_{11}, L_{22} we first find W_m for the toroids
  - To find L_{12}, we integrate $\vec{B}_1 \cdot \vec{H}_2$, over the volume of the outer toroid
  - The volumes are chosen to be everywhere the field exists – in this case, we only consider the space

in the toroids, since by Ampere's law the fields are zero outside them