Lecture 24, Mar 15, 2023

Magnetic Vector Potential

- We know $\vec{\nabla} \cdot \vec{B} = 0 \implies \oiint_S \vec{B} \cdot \mathrm{d}\vec{s} = 0$
  - Magnetic flux through a closed surface is always zero
  - Magnetic flux is denoted by Φ_m
- Since the divergence of curl is 0 we can express $\vec{B}$ as the curl of a potential $\vec{A}$
  - $\vec{B}$ has units of tesla, or webers per unit area (where weber is the unit of magnetic flux, in volt seconds)
  - $\vec{A}$ has units of tesla meters, or webers per unit length
  - Note $\vec{A}$ does not have relation to energy like V does

Definition

The magnetic vector potential $\vec{A}$ is defined such that

$$\vec{B} = \nabla \times \vec{A}$$

It is directly related to the current as

$$\vec{\nabla}^2 \vec{A} = -\mu_0 \vec{J} \implies \vec{A} = \frac{\mu_0}{4\pi} \iiint_V \frac{\vec{J}}{|\vec{R} - \vec{R}'|} \,\mathrm{d}v' = \frac{\mu_0}{4\pi} \int_L \frac{I}{|\vec{R} - \vec{R}'|} \,\mathrm{d}l'$$

which is the analogue of the Poisson equation, where μ_0 is the magnetic permeability

- Note that the magnetic vector potential is always in the same direction as $\vec{J}$
- The magnetic flux can be directly determined from the magnetic vector potential:
  - $\Phi_m = \oint_C \vec{A} \cdot \mathrm{d}\vec{l}$ where Φ_m is the magnetic flux through any surface with C as its boundary
  - This follows directly from Stokes' theorem

The Biot-Savart Law

- All magnetic phenomenon come from moving charges (in a permanent magnet, this comes from the movement of charges in the atoms)
- The Biot-Savart law relates magnetic fields to their sources
- Consider a very small bit of current (a current element or filament, which is part of a larger current loop)
  - This bit of current has position $\vec{R}'$ and creates a field at $\vec{R}$

Definition

The Biot-Savart law relates the magnetic field intensity to currents:

$$\mathrm{d}\vec{B} = \frac{\mu_0 I \,\mathrm{d}\vec{l} \times (\vec{R} - \vec{R}')}{4\pi |\vec{R} - \vec{R}'|^3}$$

- $I \,\mathrm{d}\vec{l}$ is the analogue of $\mathrm{d}Q$
- This is completely analogous to Coulomb's law, except for the cross product, which represents the right hand rule
- Given different types of current distributions we can integrate this in different ways to find $\vec{B}$:
  - Moving charge: $\vec{B} = \frac{\mu_0}{4\pi} \frac{Q\vec{u} \times (\vec{R} - \vec{R}')}{|\vec{R} - \vec{R}'|^3}$

  - Linear current loop: $\vec{B} = \frac{\mu_0}{4\pi} \int_C \frac{I \,\mathrm{d}\vec{l'} \times (\vec{R} - \vec{R}')}{|\vec{R} - \vec{R}'|^3}$
  - Surface current: $\vec{B} = \frac{\mu_0}{4\pi} \iint_S \frac{\vec{J}_s \times (\vec{R} - \vec{R}')}{|\vec{R} - \vec{R}'|^3} \,\mathrm{d}s'$
  - Volume current: $\vec{B} = \frac{\mu_0}{4\pi} \iiint_V \frac{\vec{J} \times (\vec{R} - \vec{R}')}{|\vec{R} - \vec{R}'|^3} \mathrm{d}v'$
- The Biot-Savart law can be derived from $\vec{A}$
- e.g. for a strip of length $2a$, in the x-y plane extending infinitely in the x direction, the current is $\vec{J}_s \,\mathrm{d}s' = \frac{I}{2a} \,\mathrm{d}x' \,\mathrm{d}y' \,\hat{a}_x$; find the field at $P(0, 0, z)$
  - $\mathrm{d}\vec{B} = \frac{\mu_0 \vec{J}_s \times (\vec{R} - \vec{R}') \,\mathrm{d}s'}{|\vec{R} - \vec{R}'|^3}$
  - Integrate in x', y' since those are the dimensions the strip lives in, $\mathrm{d}s' = \mathrm{d}x' \,\mathrm{d}y'$
  - $\vec{J}_s \,\mathrm{d}s' = \left(\frac{I}{2a} \vec{a}_x\right) \mathrm{d}x \,\mathrm{d}y$
  - $\vec{R} = z\vec{a}_z, \vec{R}' = x'\vec{a}_x + y'\hat{a}_y$
  - $\vec{B} = \int_{-\infty}^{\infty} \int_{-a}^{a} \frac{\mu_0 \left(\frac{I}{2a}\right) \hat{a}_x \times (-x'\hat{a}_x - y'\hat{a}_y + z\hat{a}_z)}{4\pi \left(x'^2 + y'^2 + z'^2\right)^{\frac{3}{2}}} \,\mathrm{d}y' \,\mathrm{d}x'$

    $= \frac{\mu_0 I}{8\pi a} \int_{-\infty}^{\infty} \int_{-a}^{a} \frac{-y'\hat{a}_z - z\hat{a}_y}{\left(x'^2 + y'^2 + z'^2\right)^{\frac{3}{2}}} \,\mathrm{d}y' \,\mathrm{d}x'$

    $= -\frac{\mu_0 I}{2\pi a} \tan^{-1}\left(\frac{a}{z}\right) \hat{a}_y$
    * Note we could ignore the $\hat{a}_z$ component because from symmetry and right hand rule we know the field is going to be in the $-\hat{a}_y$ direction