Lecture 22, Mar 8, 2023

Resistance and Conductance

- Recall that $\vec{J} = \sigma\vec{E}, \vec{E} = \rho\vec{J}$ where σ is the conductivity and ρ is the resistivity, with $\frac{1}{\rho} = \sigma$
- Resistance and conductance are macroscopic properties that apply to an entire piece of material rather than points within the material
- Using Ohm's law we define resistance as $R = \frac{V}{I}$, conductance as $G = \frac{I}{V} = \frac{1}{R}$
- Consider a material with conductivity σ connected to a battery with voltage V; this creates a field $\vec{E}$ that generates a current $\vec{J}$
  - We know $V = -\int \vec{E} \cdot \mathrm{d}\vec{l}$ and $I = \iint_S \vec{J} \cdot \mathrm{d}\vec{s} = \iint_S \sigma\vec{E} \cdot \mathrm{d}\vec{s}$ where S is the cross-sectional area
  - Therefore $R = \frac{V}{I} = \frac{\left|-\int \vec{E} \cdot \mathrm{d}\vec{l}\right|}{\iint_S \sigma\vec{E} \cdot \mathrm{d}\vec{s}}$
- Like capacitance, to find the resistance of any material we can assume some voltage and compute how much current this creates, and then take the ratio
- In the case of a simple conductor with uniform S and σ, $V = EL, I = E\sigma S \implies R = \frac{V}{I} = \frac{L}{\sigma S}$
  - For the uniform area conductor we know $\vec{E}$ is constant, because I is constant and therefore $\vec{J}$ must be constant, which leads to $\vec{E}$ being constant
  - This applies to e.g. a cylinder, but not a cone
  - If we don't have uniform cross-sectional area, $R = \int \frac{1}{\sigma S}\,\mathrm{d}l$ where S and σ can be functions of l
    * This works even in the case of a non-uniform electric field
    * Can be thought of as a collection of infinitesimal resistors in series
- Example: resistance of a coaxial cable filled with dielectric ε_r
  - Use Gauss's law to find $\vec{E} = \frac{\rho_{sa} a}{\varepsilon_r \varepsilon_0 r}\hat{a}_r$
  - $R = \frac{\left|\int \vec{E} \cdot \mathrm{d}\vec{l}\right|}{\iint_S \sigma\vec{E} \cdot \mathrm{d}\vec{s}} = \frac{\left|\int \frac{\rho_{sa} a}{\varepsilon_r \varepsilon_0 r}\,\mathrm{d}r\right|}{\int_0^{2\pi}\int_0^L \frac{\sigma\rho_{sa} a}{\varepsilon_r \varepsilon_0 a} a\,\mathrm{d}z\,\mathrm{d}\phi} = \frac{\ln\frac{b}{a}}{2\pi L\sigma}$
  - When we're evaluating the bottom integral to find current, we can choose any surface; usually we choose one so that $\vec{J} \cdot \mathrm{d}\vec{s}$ is easy to evaluate (e.g. a Gaussian surface)
    * We only need to integrate over a single surface, in this case a cylinder of radius a
  - We can also find this by $R = \int \frac{1}{\sigma S}\,\mathrm{d}l = \int \frac{1}{\sigma \cdot 2\pi r L}\,\mathrm{d}r$
  - Note this assumes that the electric field is uniform down the wire; if the wire were extremely long we would need to consider how $\vec{E}$ changes as you move down the wire, due to resistance of the conductor and leakage current through the dielectric

Joule's Law (Power Loss)

- Since current is the result of the electric field doing work on the electrons, the electric field has to do work to create current
- Power is lost as heat in the system
- Consider an electric field causing a current $\vec{J} = \sigma\vec{E}$ in a non-perfect dielectric; how much power does it take to sustain this current?
  - Consider a very small region; the charges move at the drift velocity
  - $\Delta P = \frac{\mathrm{d}\Delta U}{\mathrm{d}t}$ and $\Delta U = W_e$ so $\Delta P = \frac{\mathrm{d}}{\mathrm{d}t}\int \vec{F}_e \cdot \mathrm{d}\vec{l} = \frac{\mathrm{d}}{\mathrm{d}t}\int Q\vec{E} \cdot \mathrm{d}\vec{l} = \frac{\mathrm{d}}{\mathrm{d}t}\int \rho_v \Delta v\vec{E} \cdot \mathrm{d}\vec{l}$
  - $\Delta P = \int \rho_v \Delta v \vec{E} \cdot \frac{\mathrm{d}\vec{l}}{\mathrm{d}t} = \vec{E} \cdot \vec{J}\Delta v$

Definition

Joule's law:

$$P = \iiint_V \vec{E} \cdot \vec{J} \, \mathrm{d}V$$

relates energy losses in a conductor to the current and electric field in it