Lecture 17, Feb 17, 2023

Electrostatic Energy

- The work done to bring a charge from infinity in is $-\int_{-\infty}^{P_2} \vec{F}_{12} \cdot \mathrm{d}\vec{l} = -Q_1 \int_{-\infty}^{P_2} \vec{E} \cdot \mathrm{d}\vec{l} = Q_1 V(P_2)$
  - If we being in another charge we have to account for the repulsion of the additional charges already there

Definition

For a collection of point charges, the total stored energy is

$$W_e = \frac{1}{2}\sum_{i=1}^{N} Q_i V_i$$

For a continuous charge distribution this is

$$W_e = \frac{1}{2}\iiint_V \rho_v V \,\mathrm{d}V$$

where V is the potential of the total system after all the charges have been brought together

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In terms of the fields, from $\vec{\nabla} \cdot \vec{D} = \rho_v$ and $\vec{E} = -\vec{\nabla}V$, we have

$$W_e = \frac{1}{2}\iiint_V \vec{D} \cdot \vec{E}\mathrm{d}\vec{V} = \frac{1}{2}\iiint_V \varepsilon|\vec{E}|^2 \,\mathrm{d}V$$

Where the energy density is

$$W_e = \frac{1}{2}\vec{D} \cdot \vec{E} = \frac{1}{2}\varepsilon_r\varepsilon_0|\vec{E}|^2 = \frac{1}{2}\frac{|D|^2}{\varepsilon_r\varepsilon_0}$$

- The factor of $\frac{1}{2}$ accounts for duplication between charge interactions
- Consider the energy stored in a parallel plate capacitor:
  - First method: using charges
    - $W_e = \frac{1}{2}\iint \rho_s V \,\mathrm{d}S = \frac{1}{2}\iint_S \rho_s V_0 \,\mathrm{d}S = \frac{1}{2}\rho_s V_0 S = \frac{1}{2}QV_0 = \frac{1}{2}CV_0^2 = \frac{1}{2}\frac{\varepsilon_r\varepsilon_0 S}{d}V_0^2$
  - Second method: using fields
    - $W_e = \frac{1}{2}\iiint_V \varepsilon_r\varepsilon_0|\vec{E}|^2 \,\mathrm{d}V$
    - For a parallel plate capacitor $\vec{E}$ has constant magnitude $\frac{\rho_s}{\varepsilon_r\varepsilon_0}$, and the volume is Sd
    - Therefore $W_e = \frac{1}{2}\left(\frac{\rho_s}{\varepsilon_0\varepsilon_r}\right)^2 \varepsilon_r\varepsilon_0 Sd = \frac{1}{2}\frac{Q^2}{S\varepsilon_r\varepsilon_0}Sd = \frac{1}{2}\frac{Q^2 d}{\varepsilon_r\varepsilon_0 S} = \frac{1}{2}\frac{Q^2}{C}$

Important

$W_e = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV_0^2$ holds in general; we may find C from energy by $C = \frac{1}{2}\frac{Q^2}{W_e} = \frac{2W_e}{V_0^2}$