Lecture 16, Feb 15, 2023

Capacitance

Definition

Given two conductors with a potential difference, the capacitance C between them is defined as

$$C = \frac{Q}{\Delta V} = \frac{Q}{V}$$

with units of [C/V = F]

- Consider 2 conductors attached to a battery; they will have equal and opposite charges proportional to the voltage
- Let $Q = C\Delta V$; C is the *capacitance*, $C = \dfrac{Q}{\Delta V} = \dfrac{Q}{V}$[C/V = F]
  - Capacitance is a function of only the conductor geometry and the material separating them
  - A large capacitance results in large Q for a small V
  - Application notes: typically in circuits there are lots of conductors next to each other, which can introduce a parasitic capacitance; this capacitance can severely distort high frequency signals
- $C = \dfrac{Q}{\Delta V} = \dfrac{\oiint_{S+} \vec{D} \cdot \mathrm{d}\vec{S}}{\left|-\int \vec{E} \cdot \mathrm{d}\vec{l}\right|} = \dfrac{\oiint_{S+} \varepsilon_r \varepsilon_0 \vec{E} \cdot \mathrm{d}\vec{S}}{\left|-\int \vec{E} \cdot \mathrm{d}\vec{l}\right|}$
  - $S+$ is the surface that encloses the positively charged conductor but this could be another conductor as well
- Example: parallel plate capacitor filled with dielectric with relative permittivity ε_r
  - First we need to find the electric field in the dielectric: $\vec{E} = \dfrac{\rho_s}{\varepsilon_0 \varepsilon_r}$
    * We can see this from boundary conditions $D_{\text{dielectric}} - D_{\text{conductor}} = \rho_s \implies D = D_{\text{dielectric}} = \rho_s$
  - $\Delta V = \left|-\displaystyle\int \vec{E} \cdot \mathrm{d}\vec{l}\right| = Ed = \dfrac{\rho_s d}{\varepsilon_0 \varepsilon_r} = \dfrac{Qd}{\varepsilon_0 \varepsilon_r S}$
    * $Q = \rho_s S$ where S is the area of the plate, assuming uniform free charge distribution
  - Therefore $C = \dfrac{Q}{\Delta V} = \dfrac{Q}{\frac{Qd}{\varepsilon_0 \varepsilon_r S}} = \dfrac{\varepsilon_0 \varepsilon_r S}{d}$
  - This assumes uniform charge distribution (i.e. plates are effectively infinitely large), and also the insulation cannot conduct any current
- To maximize capacitance we increase ε_r or the plate area S, and making the plate spacing D as small as possible
- Example: capacitance of a spherical capacitor; inner sphere with radius a, outer sphere with radius b, dielectric with ε_r between
  - Assume a Q, find $\vec{E}$ from it, and then find ΔV
  - Assume inner shell has charge $+Q$ and outer shell has charge $-Q$
  - Apply Gauss's law: $\displaystyle\oiint_S \vec{E} \cdot \mathrm{d}\vec{S} = \frac{Q_{enc}}{\varepsilon_0 \varepsilon_r} \implies 4\pi\varepsilon_r\varepsilon_0 R^2 E_R = Q \implies \vec{E} = \frac{Q}{4\pi R^2 \varepsilon_r \varepsilon_0}\hat{a}_R$
  - Find potential by integration: $\Delta V = \left|\displaystyle\int_a^b E \,\mathrm{d}R\right| = \dfrac{Q}{4\pi\varepsilon_r\varepsilon_0}\left(\dfrac{1}{a} - \dfrac{1}{b}\right)$
  - Therefore $C = \dfrac{Q}{\Delta V} = \dfrac{4\pi\varepsilon_r\varepsilon_0 ab}{b - a}$
  - Does this make sense?
    * C is directly proportional to ε_r
    * The distance between conductors in the denominator
    * The surface area of the conductors is in the numerator

Summary

To find capacitance:

1. Assume some charge Q on the surfaces
2. Use boundary conditions and Gauss's Law to find $\vec{D}$ and $\vec{E}$
3. Find voltage $\Delta V = \left| \int \vec{E} \cdot \mathrm{d}\vec{l} \right|$
4. Calculate capacitance by $C = \dfrac{Q}{\Delta V}$